CONSTRUCTION OF THE SOLUTION OF DIRICHLET AND NEUMANN PROBLEMS FOR AN OPEN SURFACE

1970

SovietRxiv

Abstract

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UDC 517.946.9:534.26

MATHEMATICAL PHYSICS

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CONSTRUCTION OF THE SOLUTION OF DIRICHLET AND NEUMANN PROBLEMS FOR AN OPEN SURFACE

(Presented by Academician V. A. Fok, October 17, 1969)

A method is set forth for solving Dirichlet and Neumann boundary-value problems for open surfaces, to which a number of questions in the theory of diffraction, electrostatics, and the theory of analytic functions reduce. The unknown boundary values, through which the solution is expressed, are found in the form of series in constructed complete systems of orthonormal functions. To find the solution, Green' s functions of certain auxiliary domains are used. We note that the first two Green' s functions were used for the analysis of wave diffraction on open surfaces in the work ([1]), where, with their help, integral equations of the second kind were formulated for the surface current.

For definiteness, the exposition will be carried out for the inhomogeneous Helmholtz equation and zero boundary conditions. The method is simply generalized to the corresponding diffraction problems for the system of Maxwell equations.

1. Let the primary field ψ_0 be diffracted on an open surface S, whose edge is the contour L (S and L are sufficiently smooth). The sought field ψ must satisfy the equation

$$\Delta\psi + k^2\psi = F \tag{1}$$

in all space, except for the points of $S + L$; the boundary conditions on the surface S (the points of L are not counted as belonging to the surface S)*:

$$\psi^+ = \psi^- = 0 \tag{2}$$

or

$$\partial\psi^+/\partial n = \partial\psi^-/\partial n = 0. \tag{3}$$

In addition, ψ must be made subject to the Meixner condition ([2]) on the contour L, as well as to a condition at infinity.

2. Consider the Dirichlet problem ((1), (2)). Draw an auxiliary surface Σ, completing $S+L$ to a closed surface (Fig. 1). Here it is assumed that the surface $S_0' = S' + L + \Sigma$ (where S' is the part of the surface S adjacent to L) is a Lyapunov surface. All the remaining auxiliary surfaces are drawn below in the same way. The interior domain bounded by the surface $S_0 = S + L + \Sigma$ will be denoted by B_i; the complement of $B_i + S_0$ to the whole space–by B_e (Fig. 1).

Draw two more auxiliary surfaces S_1 and S_2 (Fig. 2). The domains bounded by the closed surfaces $\Sigma + L + S_1$ and $\Sigma + L + S_2$ will be denoted respectively by B_1 and B_2.

Let $G_i(x, y)$ be the Green' s function of the Dirichlet problem for the domain $B_i+\Sigma+B_1$, bounded from outside by the closed surface $S+L+S_1$, and $G_e(x, y)$ the Green' s function of the Dirichlet problem for the domain $B_e + \Sigma + B_2$,

* Here ψ^+ and $\partial\psi^+/\partial n$ are the limiting values in approaching S from the side toward which $\mathbf{n}$ is directed; ψ^-, $\partial\psi^-/\partial n$ have the analogous meaning in approaching from the opposite side. If these limiting values are equal, they will be denoted without the indices $+$, $-$.

bounded from the inside by the surface $S+L+S_2$. (The letters x and y denote points of space.)

Then, on the basis of the properties of the Green functions $G_i(x, y)$ and $G_e(x, y)$, and also of the boundary condition (2), we obtain

$$\int_\Sigma \psi(y) \frac{\partial G_i(x,y)}{\partial n_y}\, dS_y - \int_\Sigma \frac{\partial \psi(y)}{\partial n_y} G_i(x,y)\, dS_y - F_i(x) = \begin{cases} \psi(x), & x \in B_i, \\ 0, & x \in B_1, \end{cases} \tag{4}$$

$$-\int_\Sigma \psi(y) \frac{\partial G_e(x,y)}{\partial n_y}\, dS_y + \int_\Sigma \frac{\partial \psi(y)}{\partial n_y} G_e(x,y)\, dS_y - F_e(x) = \begin{cases} \psi(x), & x \in B_e, \\ 0, & x \in B_2, \end{cases} \tag{5}$$

where

$$F_i(x) = \int_{B_i} F(y) G_i(x,y)\, dV_y, \qquad F_e(x) = \int_{B_e} F(y) G_e(x,y)\, dV_y. \tag{6}$$

Introduce the vectors

$$\mathbf{U}(y) = (\psi(y), \partial\psi(y)/\partial n_y), \quad y \in \Sigma; \qquad \mathbf{F}(x_1, x_2) = (F_i(x_1), F_e(x_2)) \tag{7}$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

and the matrix

$$\Gamma(x_1, x_2; y) = \left\| \begin{array}{cc} \partial G_i(x_1, y)/\partial n_y, & -G_i(x_1, y) \\ -\partial G_e(x_2, y)/\partial n_y, & G_e(x_2, y) \end{array} \right\|. \qquad (8)$$

Then the functional equations (4′) and (5′) can be written in the form

$$\int_\Sigma \Gamma(x_1, x_2; y)\mathbf{U}(y)\, dS_y = \mathbf{F}(x_1, x_2), \qquad x_1 \in B_1,\ x_2 \in B_2, \qquad (9)$$

and the desired solution

$$(x_1, x_2) = (\psi(x_1), \psi(x_2)), \qquad x_1 \in B_i,\ x_2 \in B_e, \qquad (10)$$

can be represented in the form

$$(x_1, x_2) = \int_\Sigma \Gamma(x_1, x_2; y)\mathbf{U}(y)\, dS_y - \mathbf{F}(x_1, x_2), \qquad x_1 \in B_i,\ x_2 \in B_e. \qquad (11)$$

It is essential that, since S_0' is a Lyapunov surface, it follows (see, for example, (2)) that $\mathbf{U}(y)$, $y \in \Sigma$, belongs to the space $L_2(\Sigma)$ introduced below.

3. To solve the problem (see formula (11)), it is necessary to determine the vector-function $\mathbf{U}(y)$, $y \in \Sigma$. It can be found from the functional equation (9). For this purpose we shall use the method of V. D. Kupradze ([3]).

Fig. 1

Fig. 2

Fig. 3

Let us draw two more auxiliary surfaces S_1' and S_2' (Fig. 3) and choose on them countable everywhere dense sequences of points $\{x_i^k\}_{k=1,2,3,\ldots}$ and $\{x_e^k\}_{k=1,2,3,\ldots}$.

Fig. 3

Figure 3: Fig. 3

The surface S_1' lies between S_1 and Σ, and the surface S_2' between S_2 and Σ (Fig. 3). Denote the first and second rows of the matrix (8), regarded as vectors, respectively by $\Gamma_1(x, y)$ and $\Gamma_2(x, y)$.

Introduce the notation

$$\mathbf{K}_s(y) = \begin{cases} \Gamma_1(x_e^{(s+1)/2}, y), & s = 1, 3, 5, \dots, \\ \Gamma_2(x_i^{s/2}, y), & s = 2, 4, 6, \dots, \end{cases} \qquad y \in \Sigma. \tag{12}$$

We also define the space $L_2(\Sigma)$ of vectors $\mathbf{M}(y) = (M_1(y), M_2(y))$ given on Σ and square-summable on it. The norm in our space is defined on the basis of the scalar product

$$\langle \mathbf{M}_1(y), \mathbf{M}_2(y) \rangle = \int_\Sigma \mathbf{M}_1(y) \cdot \mathbf{M}_2^*(y)\, dS_y,$$

where $*$ denotes complex conjugation.

Theorem. *The set of vector-functions* $\{\mathbf{K}_s(y)\}$, $s = 1, 2, 3, \dots$, *is linearly independent and closed in the space* $L_2(\Sigma)$.

We introduce an orthonormal system of vector-functions obtained from the system $\{\mathbf{K}_s^*(y)\}$ by the orthogonalization process

$$\mathbf{P}_m(y) = \sum_{s=1}^{m} A_{ms} \mathbf{K}_s^*(y), \qquad m = 1, 2, 3, \dots. \tag{13}$$

The Fourier coefficient of the sought solution $\mathbf{U}(y)$ of the functional equation (9) with respect to the system of vector-functions (13) will be denoted by Φ_m:

$$\Phi_m = \langle \mathbf{U}(y), \mathbf{P}_m(y) \rangle = \sum_{s=1}^{m} A_{ms}^* \int_\Sigma \mathbf{U}(y) \cdot \mathbf{K}_s(y)\, dS_y. \tag{14}$$

As is seen from formulas (14), (12), (9),

$$\Phi_m = \sum_{s=1}^{m} A_{ms}^* \begin{cases} F_i\left(x_e^{(s+1)/2}\right), & s = 1, 3, 5, \dots, \\ F_e\left(x_i^{s/2}\right), & s = 2, 4, 6, \dots. \end{cases} \tag{15}$$

Thus the Fourier coefficients of the sought solution $\mathbf{U}(y)$ of the functional equation (9) have been found.

The series

$$\sum_{m=1}^{\infty} \Phi_m \mathbf{P}_m(y), \tag{16}$$

as the Fourier series of an element of $L_2(\Sigma)$ with respect to a complete orthonormal system of functions, converges in the norm of this space to $\mathbf{U}(y)$.

Thus, the solution $\mathbf{U}(y)$ of the functional equation (9) has been found:

$$\mathbf{U}(y) = \sum_{m=1}^{\infty} \Phi_m \mathbf{P}_m(y). \tag{17}$$

From the convergence in the $L_2(\Sigma)$-norm of the series (16) to $\mathbf{U}(y)$ there follows the uniform convergence of the expression

$$\Psi^N(x_1, x_2) = \int_\Sigma \Gamma(x_1, x_2; y) \sum_{m=1}^{N} \Phi_m \mathbf{P}_m(y)\, dS_y - F(x_1, x_2),$$

$$x_1 \in B_i, \quad x_2 \in B_e,$$

to the sought solution $\Psi(x_1, x_2)$ of the diffraction problem in any domain bounded from within by a closed surface enclosing Σ and having no common points with it, i.e. in fact in the whole space except for an arbitrarily small domain containing the (auxiliary) surface Σ.

We note that $\Psi^N(x_1, x_2)$, $x_1 \in B_i$, $x_2 \in B_e$, for all N satisfies equation (1) outside S_0 and the boundary conditions (2) on both sides of the surface S.

4. The construction of the solution of the Neumann problem ((1), (3)) differs from what has been set forth in certain details, but it likewise reduces to the solution of a functional equation of the form (9). The matrix kernel of this equation has the form (8), with the Green' s functions of the Dirichlet problems replaced by the Green' s functions of the Neumann problems for the same auxiliary domains.

Thus, the solution of boundary-value problems for open surfaces has been constructed by means of the Green' s functions of certain auxiliary domains, which are considerably simpler than the original one. At present there exists a method ([3]) for the actual construction of the Green' s functions of the auxiliary domains used above.

The author thanks V. D. Kupradze and Ya. N. Fel' d for their interest in and attention to this work.

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Received
17 X 1969

REFERENCES

[1] Ya. N. Fel' d, I. V. Sukharevskii, *Radiotekhnika i elektronika*, No. 7 (1966).
[2] Kh. Khehl, A. Maue, K. Westpfahl, *Theorie der Beugung*, Moscow, 1964.
[3] V. D. Kupradze, *Uspekhi matematicheskikh nauk*, 22, issue 2 (134) (1967).

Note: Figure translations are in progress. See original paper for figures.