ON THE INFLUENCE OF THE CORIOLIS FORCE ON THE REFLECTION OF UNSTEADY LONG WAVES

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Abstract

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ON THE INFLUENCE OF THE CORIOLIS FORCE ON THE REFLECTION OF UNSTEADY LONG WAVES

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Within the assumptions of the theory of long waves, the problem is solved of the reflection, from a rectilinear wall, of waves caused by an initial elevation of the fluid.

A homogeneous fluid of depth h fills a basin bounded by the vertical wall $y = 0$ and rotating about a vertical axis with constant angular velocity ω. In a region G of the free surface of the fluid, at the initial instant of time $t = 0$, an elevation $\zeta = \zeta_0(x, y)$ is formed. The subsequent propagation of the initial elevation of the fluid is studied.

The equations of the theory of long waves, in the commonly used notation, have the form

$$\partial u/\partial t - 2\omega v = -g\,\partial\zeta/\partial x, \qquad \partial v/\partial t + 2\omega u = -g\,\partial\zeta/\partial y, \tag{1}$$

$$\partial\zeta/\partial t = -h(\partial u/\partial x + \partial v/\partial y).$$

The initial conditions are

$$\zeta(x, y, 0) = \zeta_0(x, y), \qquad u(x, y, 0) = v(x, y, 0) = 0 \tag{2}$$

and the boundary condition is

$$v(x, 0, t) = 0. \tag{3}$$

Introducing dimensionless quantities by setting $x = hx_1,\ y = hy_1,\ u = cu_1,\ v = cv_1,\ t = (h/c)\tau,\ \zeta = h\zeta_1,\ \Omega = 2\omega c/g$, where $c = \sqrt{gh}$, we reduce the system of equations (1), the boundary and initial conditions to the form

$$\partial u_1/\partial\tau - \Omega v_1 = -\partial\zeta_1/\partial x_1, \qquad \partial v_1/\partial\tau + \Omega u_1 = -\partial\zeta_1/\partial y_1, \tag{4}$$

$$\partial\zeta_1/\partial\tau = -(\partial u_1/\partial x_1 + \partial v_1/\partial y_1);$$

$$\zeta_1(x_1, y_1, 0) = \zeta_{01}(x_1, y_1), \qquad u_1(x_1, y_1, 0) = v_1(x_1, y_1, 0) = 0 \tag{5}$$

$$v_1(x_1, 0, \tau) = 0. \tag{6}$$

Integrating system (4) with respect to τ over the limits from 0 to τ and denoting

$$\int_0^\tau u_1(x_1, y_1, \xi)\,d\xi = U_1(x_1, y_1, \tau), \qquad \int_0^\tau v_1(x_1, y_1, \xi)\,d\xi = V_1(x_1, y_1, \tau),$$

$$\int_0^\tau \zeta_1(x_1, y_1, \xi)\,d\xi = Z_1(x_1, y_1, \tau), \tag{7}$$

and also taking into account the initial conditions (5), we arrive at the system

$$\partial U_1/\partial\tau - \Omega V_1 = -\partial Z_1/\partial x_1, \qquad \partial V_1/\partial\tau + \Omega U_1 = -\partial Z_1/\partial y_1, \tag{8}$$

$$\partial Z_1/\partial\tau = \zeta_{01}(x_1, y_1) - (\partial U_1/\partial x_1 + \partial V_1/\partial y_1),$$

for which the initial conditions take the form

$$U_1(x_1, y_1, 0) = V_1(x_1, y_1, 0) = Z_1(x_1, y_1, 0) = 0 \tag{9}$$

and the boundary condition is

$$V_1(x_1, 0, \tau) = 0. \tag{10}$$

The solution of this system is sought by applying the Laplace transform in time. Denoting the images by the same letters, respectively, but without the subscript 1, we obtain for them the system

$$sU - \Omega V = -\partial Z/\partial x_1, \qquad sV + \Omega U = -\partial Z/\partial y_1,$$

$$sZ = -\zeta_{01}/s - (\partial U/\partial x_1 + \partial V/\partial y_1), \tag{11}$$

where s is the transform parameter.

From the first two equations of system (11) we have

$$U = -\frac{1}{s^2+\Omega^2}\left(s\frac{\partial Z}{\partial x_1} + \Omega\frac{\partial Z}{\partial y_1}\right), \tag{12}$$

$$V = -\frac{1}{s^2+\Omega^2}\left(s\frac{\partial Z}{\partial y_1} - \Omega\frac{\partial Z}{\partial x_1}\right). \tag{13}$$

Eliminating U and V from the third equation of system (11), we obtain

$$\frac{\partial^2 Z}{\partial x_1^2} + \frac{\partial^2 Z}{\partial y_1^2} - (s^2+\Omega^2)Z = -\frac{s^2+\Omega^2}{s^2}\zeta_{01}. \tag{14}$$

The boundary condition for equation (14) takes the form

$$s\,\partial Z/\partial y_1 - \Omega\,\partial Z/\partial x_1 = 0 \quad \text{for } y_1 = 0. \tag{15}$$

Let the initial elevation be concentrated at the point $P(x, y-b)$ and have the form

$$\zeta_0(x,y) = h\zeta_{01}(x_1,y_1) = \frac{Q}{h^2}\delta(x_1)\delta(y_1-b_1), \tag{16}$$

where $\delta(\xi)$ is the Dirac delta function.

The solution of equation (14), under the boundary condition (15) and the condition that the solution decay as $y\to\infty$, when the right-hand side of the equation is determined by equality (16), can be found by standard methods; it has the form

$$\begin{aligned} Z = \frac{Q}{4\pi h^3}\frac{s^2+\Omega^2}{s^2}\Bigg[\int_{-\infty}^{\infty} \exp\left[-(b_1-y_1)\sqrt{s^2+\Omega^2+\alpha^2}\right]\exp[i\alpha x_1]\,\frac{d\alpha}{\sqrt{s^2+\alpha^2+\Omega^2}} \\ + \int_{-\infty}^{\infty}\frac{s\sqrt{s^2+\alpha^2+\Omega^2}-\Omega\alpha i}{s\sqrt{s^2+\alpha^2+\Omega^2}+\Omega\alpha i} \\ \times \exp\left[-(b_1+y_1)\sqrt{s^2+\alpha^2+\Omega^2}\right]\frac{\exp[i\alpha x_1]}{\sqrt{s^2+\alpha^2+\Omega^2}}\,d\alpha\Bigg], \end{aligned} \tag{17}$$

whence, according to (12) and (13), we obtain

$$U = -\frac{Q}{4\pi h^3}\frac{1}{s^2}\left[\int_{-\infty}^{\infty}\frac{i\alpha s+\Omega\sqrt{s^2+\alpha^2+\Omega^2}}{\sqrt{s^2+\alpha^2+\Omega^2}}\exp\left[-(b_1-y_1)\sqrt{s^2+\alpha^2+\Omega^2}\right]\right.$$
$$\times\exp[i\alpha x_1]\,d\alpha-\int_{-\infty}^{\infty}\frac{s\sqrt{s^2+\alpha^2+\Omega^2}-\Omega\alpha i}{s\sqrt{s^2+\alpha^2+\Omega^2}+\Omega\alpha i}(i\alpha s-\Omega\sqrt{s^2+\alpha^2+\Omega^2})$$
$$\left.\times\frac{\exp\left[-(b_1+y_1)\sqrt{s^2+\alpha^2+\Omega^2}\right]}{\sqrt{s^2+\alpha^2+\Omega^2}}\exp[i\alpha x_1]\,d\alpha\right], \tag{18}$$

$$V = -\frac{Q}{4\pi h^3}\frac{1}{s^2}\left[\int_{-\infty}^{\infty}\frac{s\sqrt{s^2+\alpha^2+\Omega^2}-\Omega\alpha i}{\sqrt{s^2+\alpha^2+\Omega^2}}\exp\left[-(b_1-y_1)\sqrt{s^2+\alpha^2+\Omega^2}\right]\right.$$
$$\times\exp[i\alpha x_1]\,d\alpha-\int_{-\infty}^{\infty}\frac{s\sqrt{s^2+\alpha^2+\Omega^2}-\Omega\alpha i}{\sqrt{s^2+\alpha^2+\Omega^2}}$$
$$\left.\times\exp\left[-(b_1+y_1)\sqrt{s^2+\alpha^2+\Omega^2}\right]\exp[i\alpha x_1]\,d\alpha\right]. \tag{19}$$

Performing the inverse transforms, taking into account the equalities

$$\frac{s^2+\Omega^2}{s^2}\frac{s\sqrt{s^2+\alpha^2+\Omega^2}-\Omega\alpha i}{s\sqrt{s^2+\alpha^2+\Omega^2}+\Omega\alpha i}=1-\frac{\Omega^2}{s^2}+\frac{2\Omega^2}{s^2+\alpha^2}-2\Omega\alpha i\frac{\sqrt{s^2+\alpha^2+\Omega^2}}{s(s^2+\alpha^2)},$$

$$\frac{1}{s^2}\frac{s\sqrt{s^2+\alpha^2+\Omega^2}-\Omega\alpha i}{s\sqrt{s^2+\alpha^2+\Omega^2}+\Omega\alpha i}(i\alpha s-\Omega\sqrt{s^2+\alpha^2+\Omega^2})=\frac{i\alpha}{s}+\Omega\frac{\sqrt{s^2+\alpha^2+\Omega^2}}{s^2}$$
$$-2\Omega\frac{\sqrt{s^2+\alpha^2+\Omega^2}}{s^2+\alpha^2}+2\Omega^2\frac{i\alpha}{s(s^2+\alpha^2)},$$

we obtain that the final result can be written in the form

$$\zeta=\frac{Q}{2\pi h^2}\left[\left(\frac{\partial^2 M}{\partial\tau^2}+\Omega^2 M\right)_{y=y_1}+\left(\frac{\partial^2 M}{\partial\tau^2}-\Omega^2 M\right)_{y=-y_1}+2\Omega\left(\frac{\partial^2 N}{\partial x_1\partial y_1}+\Omega\frac{\partial N}{\partial\tau}\right)\right]; \tag{20}$$

$$u=-\frac{Qc}{2\pi h^3}\left[\left(\frac{\partial^2 M}{\partial\tau\partial x_1}+\Omega\frac{\partial M}{\partial y_1}\right)_{y=y_1}+\left(\frac{\partial^2 M}{\partial\tau\partial x_1}-\Omega\frac{\partial M}{\partial y_1}\right)_{y=-y_1}+2\Omega\left(\frac{\partial^2 N}{\partial\tau\partial y_1}+2\Omega\frac{\partial N}{\partial x_1}\right)\right]; \tag{21}$$

$$v = -\frac{Qc}{2\pi h^3}\left[\left(\frac{\partial^2 M}{\partial\tau\partial y_1} - \Omega\frac{\partial M}{\partial x_1}\right)_{y=y_1} + \left(\frac{\partial^2 M}{\partial\tau\partial y_1} + \Omega\frac{\partial M}{\partial x_1}\right)_{y=-y_1}\right], \qquad (22)$$

where

$$M(x_1, y, \tau) = \frac{1}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\exp[s\tau]}{s}\,ds\int_{-\infty}^{\infty}\exp\left[-(b_1 - y)\sqrt{s^2+\alpha^2+\Omega^2}\right]\times\frac{\exp[i\alpha x_1]\,d\alpha}{\sqrt{s^2+\alpha^2+\Omega^2}}, \qquad (23)$$

$$N(x_1, y_1, \tau) = \frac{1}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\exp[s\tau]\,ds\int_{-\infty}^{\infty}\exp\left[-(b_1 + y_1)\sqrt{s^2+\alpha^2+\Omega^2}\right]\times\frac{\exp[i\alpha x_1]\,d\alpha}{(s^2+\alpha^2)\sqrt{s^2+\alpha^2+\Omega^2}}. \qquad (24)$$

The integral expressions (23) and (24) can be substantially simplified. Integral (23), with the aid of the known formula (1) for the integral representation of the Macdonald function $K(z)$, is reduced to the form

$$M = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}K_0\left[\gamma\sqrt{x_1^2+(b_1-y_1)^2}\right]e^{s\tau}\frac{ds}{s}, \qquad \gamma = \sqrt{s^2+\Omega^2}, \qquad (25)$$

after which it is established that

$$M_* = 2H(\tau - r_1)\int_0^{\sqrt{\tau^2-r_1^2}}\frac{\cos\Omega\eta}{\sqrt{\eta^2+r_1^2}}\,d\eta, \qquad r_1^2 = x_1^2 + (y_1 - b_1)^2, \qquad (26)$$

where $H(\tau - r_1)$ is the Heaviside function.

To transform the integral N, we use a theorem from the theory of Fourier transforms (2)

$$\int_{-\infty}^{\infty}F(u)G(u)e^{-ixu}\,du = \int_{-\infty}^{\infty}g(\xi)f(x-\xi)\,d\xi,$$

where $F(u)$ and $G(u)$ are the Fourier transforms, respectively, of the functions $f(\tau)$ and $g(\tau)$. Taking $F(u) = 1/(s^2+u^2)$,

$$G(u) = \exp\left[-(b_1 - y_1)\sqrt{s^2+\alpha^2+\Omega^2}\right]/\sqrt{s^2+\alpha^2+\Omega^2},$$

we obtain

$$N = \frac{1}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{s\tau}\frac{ds}{s}\int_0^\infty K_0\left(\gamma\sqrt{(x_1-\xi)^2+(y_1+b_1)^2}\right)e^{-s\xi}\,d\xi +$$

$$+\frac{1}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{s\tau}\frac{ds}{s}\int_0^\infty K_0\left(\gamma\sqrt{(x_1+\xi)^2+(y_1+b_1)^2}\right)e^{-s\xi}\,d\xi. \qquad (27)$$

Differentiating N with respect to τ and interchanging the order of integration (the legitimacy of these operations can be rigorously proved), we arrive at the expression

$$\frac{\partial N}{\partial \tau} = \frac{1}{4\pi i}\int_0^\infty d\xi\int_{\sigma-i\infty}^{\sigma+i\infty} e^{s(\tau-\xi)}K_0\left(\gamma\sqrt{(x_1-\xi)^2+(y_1+b_1)^2}\right)\,d\xi +$$

$$+\frac{1}{4\pi i}\int_0^\infty d\xi\int_{\sigma-i\infty}^{\sigma+i\infty} e^{s(\tau-\xi)}K_0\left(\gamma\sqrt{(x_1+\xi)^2+(y_1+b_1)^2}\right)\,d\xi.$$

Here both integrations can be carried out, the inner one by means of formula (26), and the outer one by an elementary change of variables. As a result we find

$$N = \frac{1}{\Omega}H(\tau-r_2)\int_{r_2}^{\tau}\frac{\lambda}{\lambda^2-x_1^2}\sin\Omega\sqrt{\lambda^2-r_2^2}\,d\lambda =$$

$$= \frac{1}{\Omega}H(\tau-r_2)\int_{y_1+b_1}^{\sqrt{\tau^2-x_1^2}}\sin\Omega\sqrt{\mu^2-(y_1+b_1)^2}\,\frac{d\mu}{\mu}, \qquad (28)$$

where $r_2^2 = x_1^2 + (y_1+b_1)^2$.

Formulas (20)–(22), together with (26) and (28), give the final answer to the problem. For example, for the elevation $\zeta(x, y, t)$, on the basis of these formulas we obtain

$$\zeta(x,y,t) = \frac{Q}{\pi h^2}\left\{\frac{\delta(\tau - r_1)}{\sqrt{\tau^2 - r_1^2}} - H(\tau - r_1)\left[\tau\left(\frac{\Omega\sin\Omega\sqrt{\tau^2 - r_1^2}}{\tau^2 - r_1^2} + \frac{\cos\Omega\sqrt{\tau^2 - r_1^2}}{(\tau^2 - r_1^2)^{3/2}}\right)\right.\right.$$
$$\left. + \Omega^2 \int_0^{\sqrt{\tau^2 - r_1^2}} \frac{\cos\Omega\eta}{\sqrt{\eta^2 + r_1^2}}\, d\eta\right] + \frac{\delta(\tau - r_2)}{\sqrt{\tau^2 - r_2^2}}$$
$$- H(\tau - r_2)\left[\tau\left(\frac{\Omega\sin\Omega\sqrt{\tau^2 - r_2^2}}{\tau^2 - r_2^2} + \frac{\cos\Omega\sqrt{\tau^2 - r_2^2}}{(\tau^2 - r_2^2)^{3/2}}\right)\right.$$
$$\left. - \Omega^2 \int_0^{\sqrt{\tau^2 - r_2^2}} \frac{\cos\Omega\eta}{\sqrt{\eta^2 + r_2^2}}\, d\eta\right]$$
$$+ 2\Omega H(\tau - r_2)\left[\frac{x_1(y_1 + b_1)\cos\Omega\sqrt{\tau^2 - x_1^2 - (y_1 + b_1)^2}}{(\tau^2 - x_1^2)\sqrt{\tau^2 - x_1^2 - (y_1 + b_1)^2}}\right.$$
$$\left.\left. + \frac{\tau}{\tau^2 - x_1^2}\sin\Omega\sqrt{\tau^2 - r_2^2}\right]\right\}. \tag{29}$$

The first four terms of expression (29) correspond to the direct wave, and the subsequent ones to waves reflected from the boundary.

Analogous expressions for the direct wave were obtained by Chambers ([3]), who solved the problem for an unbounded fluid in a somewhat different formulation.

We note that two terms of expression (29), represented in integral form, correspond to the so-called "geostrophic elevation" (as $\tau \to \infty$, the integrals are represented by the Macdonald function). On the boundary, the sum of the elevations represented by these terms is zero. The fundamental solution obtained can be used in solving the corresponding problems on the reflection of initial elevations distributed over some area. Then the solution obtained will not have an infinite discontinuity at the front, since the maximum order of the singularity $(\tau^2 - r_1^2)^{-3/2}$ will be smoothed by twofold integration. This solution, by convolution, can also be used for solving problems on the reflection of waves from pressures applied to the surface of the fluid over some time interval.

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Note: Figure translations are in progress. See original paper for figures.