POTENTIAL THEORY FOR LYAPUNOV-DINI DOMAINS

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Abstract

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POTENTIAL THEORY FOR LYAPUNOV-DINI DOMAINS

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Definition. We shall say that a surface $S \in A_{\varphi}^{(1)}$ if $S \in C^{(1)}$ and, for any x and $y \in S$, $\theta \leq a\varphi(r)$, where θ is the angle between the normals to the surface at the points x and y, $r = |x - y|$, and $a > 0$ is a fixed constant (see, for example, ([1])).

B. N. Khimchenko and the author proved the following theorem:

Theorem 1. Let $S \in A_{\varphi}^{(1)}$ and let it bound a domain D. Then the Neumann problem:

$$\Delta U = 0 \quad \text{in } D, \qquad \Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2},$$

$$\left.\frac{\partial U}{\partial n}\right|_S = 0, \qquad U \in C^{(0)}(\overline{D}) \cap C^{(2)}(D),$$

has only the trivial solution (see ([2])).

In the proof the author constructed a harmonic function $v(x_1, \rho)$ such that $\partial v/\partial x_1|_{x_1=\rho=0} > 0$, while on the lateral surface of the paraboloid of revolution $x_1 = \rho\varphi(\rho)$ the function $v(x_1, \rho) \leq 0$, $v(0, 0) = 0$, where $\rho^2 = \sum_{i=2}^{m} x_i^2$; for the construction the Poisson formula for the ball was used.

Relying on this theorem, the author transferred the principal results of potential theory to surfaces of class $A_{\varphi}^{(1)}$. The proofs of all the following theorems are carried out essentially in the same way as for Lyapunov surfaces, provided the following facts are used:

1) Let $S \in A_{\varphi}^{(1)}$; then there exists a constant $d > 0$ such that for an arbitrary point $x \in S$ there exists a Lyapunov sphere.
2) If at a point $O \in S$ a local coordinate system is introduced, with the direction of the outward normal coinciding with the axis OX_m, then for

points of the surface $x \in S \cap K(0,d)$, where $K(0,d)$ is a Lyapunov sphere with center at O,

$$|\cos(\nu, OX_k)| \leq \sqrt{3}\, a\varphi(r), \quad k = 1, 2, \dots, m-1;$$

ν is the normal at the point x;

$$\cos(\nu, OX_m) \geq \tfrac{1}{2}; \qquad |x_m| \leq ar\varphi(r), \qquad |\cos(\nu, r)| \leq c(a,m)\varphi(r)$$

(see, for example, ([3])).

Let

$$W(x) = \int_S \sigma(\xi) \frac{\partial}{\partial \nu} \left(\frac{1}{r^{m-2}} \right) dS, \qquad \sigma(\xi) \in C^{(0)}(S), \quad r = |x - \xi|, \quad S \in A_\varphi^{(1)}.$$

Theorem 2. $W(x)$ exists for $x \in S$, is a continuous function on S, and the following relations hold:

$$W_i(x_0) = \frac{(m-2)|S_1|}{2}\sigma(x_0) + \overline{W(x_0)},$$

$$W_e(x_0) = -\frac{(m-2)|S_1|}{2}\sigma(x_0) + \overline{W(x_0)},$$

where W_i and W_e are the limiting values of $W(x)$ as $x \to x_0 \in S$, respectively from the inside and from the outside; $\overline{W(x_0)} = W(x_0)$, $x_0 \in S$, and the convergence is uniform with respect to $x_0 \in S$.

Theorem 3. Let

$$|\sigma(x) - \sigma(x')| \leq A|x - x'|, \qquad A = \text{const}; \quad x, x' \in S; \quad S \in A_\varphi^{(1)}.$$

If the potential W has one of the normal deriv-

$$\frac{\partial W}{\partial n_e}, \quad \frac{\partial W}{\partial n_i};$$

at the point $x_0 \in S$, it also has the other normal derivative, and

$$\frac{\partial W}{\partial n_e} = \frac{\partial W}{\partial n_i}\bigg|_{x=x_0}.$$

Theorem 4. Suppose that the conditions of the preceding theorem are satisfied and

$$\left|\int_0^{2\pi} (\sigma(x) - \sigma(x_0))\, d\varphi\right| \le a\rho\varphi(\rho),$$

where $|x - x_0| = \sqrt{\rho^2 + z^2}$ in the local coordinate system; then W has a normal derivative at the point x_0.

Let

$$V(x) = \int_S \frac{\mu(\xi)}{r^{m-2}}\, dS, \qquad r = |x - \xi|.$$

Theorem 5. If $S \in A_\varphi^{(1)}$, $\mu \in C^{(0)}(S)$, then on the surface S the simple-layer potential has the normal derivative

$$\frac{\partial V}{\partial n_i} = \frac{(m-2)|S_1|}{2}\mu(x_0) + \frac{\overline{\partial V}}{\partial n},$$

$$\frac{\partial V}{\partial n_e} = -\frac{(m-2)|S_1|}{2}\mu(x_0) + \frac{\overline{\partial V}}{\partial n},$$

where $\partial V/\partial n_i$, $\partial V/\partial n_e$ are the limiting values of $\partial V/\partial n$, respectively from inside and outside S, and the convergence is uniform with respect to $x_0 \in S$;

$$\frac{\overline{\partial V}}{\partial n} = \int_S \mu(\xi) \frac{\partial}{\partial n}\left(\frac{1}{r^{m-2}}\right) dS.$$

Theorem 6. If $S \in A_\varphi^{(1)}$, $|\mu(x) - \mu(x')| \le \psi(|x - x'|)$,

$$\int_0^1 \frac{\psi(x)}{x}\, dx < \infty,$$

then the derivatives $\partial V/\partial x_1, \dots, \partial V/\partial x_m$ are uniformly continuous functions both in the interior and in the exterior domain.

Theorem 7. Let $\overline{\partial V}/\partial n = F(x)$, $\mu \in C^{(0)}(S)$, $S \in A_{\varphi_1}^{(1)}$,

$$\int_0^1 \frac{dt}{t} \int_0^t \frac{\varphi_1(x)}{x}\, dx < \infty;$$

then

$$|F(x) - F(x')| \le B\psi(|x - x'|), \qquad x, x' \in S;$$

$$B = \text{const}, \qquad \int_0^1 \frac{\psi(x)}{x}\, dx < \infty.$$

For the limiting values $W(x)$, $V(x)$, when $\mu, \sigma \in L_1(S)$, exactly the same theorems are valid as in the case of Lyapunov surfaces, if one uses the strengthened theorem of F. Riesz:

Theorem 8. If m_0 is a Lebesgue point of the summable function $\mu(x)$ on S, then

$$\int_{(m_0,\delta)} \frac{|\mu-\mu_0|}{\rho^{m-1}}\varphi(\rho)\,dS \xrightarrow[\delta\to 0]{} 0, \qquad (m_0,\delta) = K(m_0,\delta)\cap S.$$

It is now clear that the first and second boundary-value problems for the Laplace equation with continuous boundary data on the surface $S \in A_\varphi^{(1)}$ can be reduced to integral equations, and the following integral operators are obtained:

$$(Ku)(x) = \int_\Omega K(x,\xi)u(\xi)\,d\xi, \qquad \Omega \subset E_{m-1},$$

$$K(x,\xi) = A(x,\xi)\varphi(r)/r^{m-1}, \qquad |A(x,\xi)| \le C.$$

Theorem 9. The integral operator K is defined on the entire space $L_2(\Omega)$ and is bounded in it; moreover, it is completely continuous in $L_2(\Omega)$.

Theorem 10. The integral operator K is completely continuous in the space $C^{(0)}(\Omega)$ of functions continuous in Ω, if $A(x,\xi)$ is continuous in Ω.

Hence we immediately obtain:

Theorem 11. If $S \in A_\varphi^{(1)}$, then the interior and exterior Dirichlet and Neumann problems are solvable for arbitrary continuous boundary conditions, and the solutions can be represented, respectively, in the form of double- and single-layer potentials.

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Note: Figure translations are in progress. See original paper for figures.