SovietRxiv

Soviet-era science, translated into English

ON THE CONDITIONS FOR APPLYING EULER'S METHOD TO THE STUDY OF THE STABILITY OF DEFORMATION OF NONLINEARLY ELASTIC BODIES UNDER FINITE SUBCRITICAL DEFORMATIONS

1970

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-197001.98404

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

UDC 539.3

THEORY OF ELASTICITY

A. N. GUZ

ON THE CONDITIONS FOR APPLYING EULER' S METHOD TO THE STUDY OF THE STABILITY OF DEFORMATION OF NONLINEARLY ELASTIC BODIES UNDER FINITE SUBCRITICAL DEFORMATIONS

(Presented by Academician L. I. Sedov on February 19, 1970)

Within the framework of the theory of small subcritical deformations, under certain simplifications, sufficient conditions for the application of Euler's method in investigating the stability of deformation of elastic bodies were obtained in ([1]). These conditions consist in requiring that the external forces be conservative.

In the present paper we obtain sufficient conditions for applying Euler' s method within the framework of the theory of finite subcritical deformations for an isotropic nonlinear elastic body with an arbitrary form of potential.

Let us refer a three-dimensional body to Lagrangian coordinates, which before deformation coincide with Cartesian coordinates, and assume that the stability of deformation can be investigated from the linearized equations. Quantities of the basic subcritical state will be denoted by the subscript zero, and perturbation quantities without a subscript. In the case of finite subcritical deformations, the basic linearized equations ([3]) may be given the form

$$\left[\sigma_{in}^{*}\left(\delta_{nm}+u_{m,n}^{0}\right)+\sigma_{in}^{*0}u_{m,n}\right]_{,i}+X_{m}^{*}-\rho^{*}\ddot{u}_{m}=0,\qquad m=1,2,3. \tag{1}$$

The boundary conditions in stresses on the part of the body surface S_1 and in displacements on the part of the surface S_2 we write ([3]) in the form

$$N_i\left[\sigma_{in}^{*}\left(\delta_{nm}+u_{m,n}^{0}\right)+\sigma_{in}^{*0}u_{m,n}\right]_{S_1}=P_m^{*};\qquad u_m\big|_{S_2}=0. \tag{2}$$

Here σ_{in}^{*} are perturbations of the generalized stresses; X_m^{*} are perturbations of body forces, ρ^{*} is the material density referred to the unit volume of the body before deformation; N_i are unit normals to the body surface before deformation; P_m^{*} are perturbations of the components of the surface forces acting on the body

SovietRxiv

after deformation, but referred to the unit area before deformation; the volume of the body V and the surface $S = S_1 + S_2$ should be understood as before deformation because of the use of Lagrangian coordinates.

Consider an isotropic nonlinear elastic body with an arbitrary form of potential $\Phi = \Phi(I_1', I_2', I_3')$, depending on three algebraic invariants. The generalized stresses $\sigma_{ij}^{*'}$ are determined by the formula

$$\sigma_{ij}^{*'} = \frac{1}{2}\left(\frac{\partial}{\partial\varepsilon_{ij}'} + \frac{\partial}{\partial\varepsilon_{ji}'}\right)\Phi(I_1', I_2', I_3'). \tag{3}$$

Here $2\varepsilon_{ij}' = u_{i,j}' + u_{j,i}' + u_{s,i}'u_{s,j}'$ are the components of the Green strain tensor; the prime index has been introduced so as not to identify the quantities with their perturbations.

Linearizing relation (2), we obtain

$$\sigma_{ij}^{*} = \lambda_{ij\alpha\beta}u_{\alpha,\beta}, \tag{4}$$

where the notation

$$\begin{aligned}\lambda_{ij\alpha\beta} = {} & (\delta_{ij}\partial^2\Phi^0/\partial I_1^0\partial I_k^0 + 2\varepsilon_{ij}^0\partial^2\Phi^0/\partial I_2^0\partial I_k^0 + 3\varepsilon_{it}^0\varepsilon_{tj}^0\partial^2\Phi^0/\partial I_3^0\partial I_k^0)[\delta_{k1}(\delta_{\alpha\beta} + u_{\alpha\beta}^0) \\ & + (\delta_{k2}\varepsilon_{nm}^0 - \tfrac{3}{2}\delta_{k3}\varepsilon_{mp}^0\varepsilon_{pn}^0)(\delta_{n\alpha}\delta_{m\beta} + \delta_{m\alpha}\delta_{n\beta} + u_{\alpha,m}^0\delta_{n\beta} + u_{\alpha,n}^0\delta_{m\beta})] \\ & + (\delta_{\alpha i}\delta_{\beta j} + \delta_{\alpha j}\delta_{\beta i} + u_{\alpha,j}^0\delta_{\beta i} + u_{\alpha,i}^0\delta_{\beta j})\partial\Phi^0/\partial I_2^0 \\ & + \tfrac{3}{2}\,[\varepsilon_{it}^0(\delta_{t\alpha}\delta_{j\beta} + \delta_{j\alpha}\delta_{t\beta} + u_{\alpha,j}^0\delta_{t\beta} + u_{\alpha,t}^0\delta_{j\beta}) \\ & + \varepsilon_{tj}^0(\delta_{i\alpha}\delta_{t\beta} + \delta_{t\alpha}\delta_{i\beta} + u_{\alpha,t}^0\delta_{i\beta} + u_{\alpha,i}^0\delta_{t\beta})]\partial\Phi^0/\partial I_3^0.\end{aligned} \tag{5}$$

Let us note that, in the general case, $\lambda_{ij\alpha\beta} = \lambda_{ij\alpha\beta}(x_1, x_2, x_3)$. Substituting (4) into (1) and (2), we obtain

$$[\lambda_{in\alpha\beta}(\delta_{nm} + u_{m,n}^0)u_{\alpha,\beta} + \sigma_{in}^{*0}u_{m,n}]_{,i} + X_m^* - \rho^*\ddot{u}_m = 0, \tag{6}$$

$$N_i[\lambda_{in\alpha\beta}(\delta_{nm} + u_{m,n}^0)u_{\alpha,\beta} + \sigma_{in}^{*0}u_{m,n}]_{S_1} = P_m^*, \qquad u_m|_{S_2} = 0. \tag{7}$$

The problem of stability of deformation under the dynamic approach is reduced to the study of small oscillations about the equilibrium position, i.e., to determining the minimum values of σ_{in}^{*0} for which we obtain solutions of the boundary-value problem (6)–(7) that grow in time; here, in the quantities u_i, X_m^*, and P_m^*, we single out the factor $\exp i\omega t$.

SovietRxiv

Euler’ s method (the static approach) consists in finding the conditions under which, along with the unperturbed equilibrium form, adjacent perturbed equilibrium forms exist, i.e., in determining the minimum values of σ_{in}^{*0} for which the solution of the following **boundary-value problem** is nonunique:

$$[\lambda_{in\alpha\beta}(\delta_{nm} + u_{m,n}^0)u_{\alpha,\beta} + \sigma_{in}^{*0}u_{m,n}]_{,i} + X_m^* = 0, \tag{8}$$

$$N_i[\lambda_{in\alpha\beta}(\delta_{nm} + u_{m,n}^0)u_{\alpha,\beta} + \sigma_{in}^{*0}u_{m,n}]_{S_1} = P_m^*, \qquad u_m|_{S_2} = 0. \tag{9}$$

Repeating the course of reasoning of ([1]), we arrive at the conclusion that a sufficient condition for the applicability of Euler’ s method, also within the framework of the theory of finite precritical deformations, is the self-adjointness of the boundary-value problem ([2]).

Let $u_i^{(1)}$ and $u_i^{(2)}$ be the components of two arbitrary twice-differentiable vectors satisfying the boundary conditions (9); in addition, by the indices 1 and 2 we shall denote the corresponding quantities of the perturbations of the surface and body forces. We form the condition of self-adjointness

$$\Delta = \int_V [u_m^{(1)} L_m(u_1^{(2)}, u_2^{(2)}, u_3^{(2)}) - u_m^{(2)} L_m(u_1^{(1)}, u_2^{(1)}, u_3^{(1)})]\, dV = 0, \tag{10}$$

which must hold for any $u_i^{(1)}$ and $u_i^{(2)}$ satisfying the conditions (9). Here L_m denotes the equations of system (8).

Applying the Gauss-Ostrogradsky formula and taking into account the boundary conditions (9) and the symmetry of the generalized stresses, after a number of transformations we obtain

$$\begin{aligned}\Delta = \iint_{S_1} (u_m^{(1)} P_m^{*(2)} - u_m^{(2)} P_m^{*(1)})\, dS + \int_V (u_m^{(1)} X_m^{*(2)} - u_m^{(2)} X_m^{*(1)})\, dV \\ - \int_V [\lambda_{in\alpha\beta}(\delta_{nm} + u_{m,n}^0) - \lambda_{\beta nmi}(\delta_{n\alpha} + u_{\alpha,n}^0)] u_{m,i}^{(1)} u_{\alpha,\beta}^{(2)}\, dV.\end{aligned} \tag{11}$$

Since the first two integrals in (11) depend on the applied forces, and the third on the form of the elastic potential, it follows from (10) and (11), by setting separately to zero the indicated parts of expression (11), that we obtain sufficient conditions for the applicability of Euler’ s method within the framework of the theory of finite precritical-

···ical deformations in the form

$$\iint_{S_1} \left(u_m^{(1)} P_m^{*(2)} - u_m^{(2)} P_m^{*(1)}\right) dS + \int_V \left(u_m^{(1)} X_m^{*(2)} - u_m^{(2)} X_m^{*(1)}\right) dV = 0, \tag{12}$$

SovietRxiv

$$\int_V \left[\lambda_{in\alpha\beta}\left(\delta_{nm}+u^0_{m,n}\right)-\lambda_{\beta nmi}\left(\delta_{n\alpha}+u^0_{\alpha,n}\right)\right]u^{(1)}_{m,i}u^{(2)}_{\alpha,\beta}\,dV=0. \qquad (13)$$

Condition (12) is satisfied in the particular case of "dead" external loads ($P^*_m = 0,\ X^*_m = 0$), and in the more general case of conservative loads. Thus, condition (12), obtained within the theory of finite precritical deformations, coincides with the known condition (1), obtained within the theory of small precritical deformations with certain simplifications.

Let us analyze condition (13), which, together with condition (12), is sufficient. Owing to the arbitrariness of the choice of $u^{(1)}_i$ and $u^{(2)}_i$, we may set

$$\delta_{i\alpha\beta m}=\lambda_{ij\alpha\beta}\left(\delta_{mj}+u^0_{m,j}\right)-\lambda_{\beta jmi}\left(\delta_{\alpha j}+u^0_{\alpha,j}\right)=0. \qquad (14)$$

Condition (14) is satisfied for the Treloar and Mooney potentials, and also for the harmonic potential.

Consider the simplifications that arise if the precritical deformations are regarded as small and the precritical state is determined by the classical theory of elasticity. Expression (5) takes the form

$$\lambda_{ij\alpha\beta}=\left(\delta_{ij}\frac{\partial^2\Phi^0}{\partial I^0_1\partial I^0_k}+2\varepsilon^0_{ij}\frac{\partial^2\Phi^0}{\partial I^0_2\partial I^0_k}-3\varepsilon^0_{it}\varepsilon^0_{tj}\frac{\partial^2\Phi^0}{\partial I^0_3\partial I^0_k}\right)\left(\delta_{\alpha\beta}\delta_{k1}+2\varepsilon^0_{\alpha\beta}\delta_{k2}+3\varepsilon^0_{\alpha p}\varepsilon^0_{p\beta}\delta_{k3}\right)$$
$$+\left(\delta_{\alpha i}\delta_{rj}+\delta_{\alpha j}\delta_{\beta i}\right)\frac{\partial\Phi^0}{\partial I_2}+\left(\delta_{j\beta}\varepsilon^0_{i\alpha}+\delta_{j\alpha}\varepsilon^0_{i\beta}+\delta_{i\alpha}\varepsilon^0_{j\beta}+\delta_{i\beta}\varepsilon^0_{\alpha j}\right)\frac{\partial\Phi^0}{\partial I_3},\qquad 2\varepsilon^0_{ij}=u^0_{i,j}+u^0_{j,i}. \qquad (15)$$

In this case condition (14), taking into account the symmetry of the stresses, can be put in the form

$$\lambda_{\alpha\beta mi}=\lambda_{mi\alpha\beta}. \qquad (16)$$

From expression (15), by direct verification one can see that conditions (16) are satisfied for any form of the elastic potential.

Thus, within the theory of small precritical deformations, when the precritical state is determined by a geometrically linear theory, for a body with a potential of arbitrary form, the only sufficient condition for the application of Euler' s method is the conservativeness of the external loads, which coincides with the results of [1].

It should be noted that only sufficient conditions have been considered here, and their nonfulfillment does not mean that Euler' s method cannot be applied.

Institute of Mechanics
Academy of Sciences of the Ukrainian SSR
Kiev

Received
26 IX 1969

CITED LITERATURE

[1] V. V. Bolotin, *Nonconservative Problems of the Theory of Elastic Stability*, Moscow, 1961.
[2] S. G. Mikhlin, *Problems of the Minimum of a Quadratic Functional*, Moscow, 1952.
[3] V. V. Novozhilov, *Foundations of the Nonlinear Theory of Elasticity*, 1948.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.