

# ON THE BEHAVIOR OF THE ENERGY OF A SOLUTION OF THE WAVE EQUATION IN AN UNBOUNDED DOMAIN FOR LARGE TIMES

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**Abstract**

**Full Text**

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**MATHEMATICS**

**A. A. ARSEN' EV**

**ON THE BEHAVIOR OF THE ENERGY OF A SOLUTION OF THE WAVE EQUATION IN AN UNBOUNDED DOMAIN FOR LARGE TIMES**

*(Presented by Academician A. N. Tikhonov, February 3, 1970)*

1. Let  $R_N$  be  $N$ -dimensional Euclidean space,  $N \geq 3$ , and let  $\Omega$  be an open domain in  $R_N$  with boundary of class  $A^{(1,\alpha)}$ ,  $\alpha > 0$ . We shall assume that  $\Omega$  contains the exterior of some ball. In the cylinder  $\Omega \times [0, \infty)$  consider the mixed problem

$$\frac{\partial^2 v}{\partial t^2} - \Delta v = F(x, t), \quad x \in \Omega, \quad t > 0, \quad v(x, t) \in \dot{W}_2^1(\Omega),$$

$$v(x, +0) = v_t(x, +0) = 0, \quad x \in \Omega. \tag{1}$$

Let  $v(x, t)$  be the solution of problem (1); call the energy  $E(t)$  the integral

$$E(t) = \int_{\Omega} (|v_t(x, t)|^2 + |\nabla_x v(x, t)|^2) dx.$$

The main result of this work is formulated in Theorems 1 and 2; it consists in calculating the asymptotics of the function  $E(t)$  as  $t \rightarrow \infty$  for certain almost periodic functions  $F(x, t)$ .

2. Let  $u(x, k)$  be the solution of the scattering problem for the domain  $R_N \setminus \Omega$ :

$$-\Delta u = \lambda u, \quad x \in \Omega, \quad \lambda = k^2, \quad k \in R_N, \quad u(x, k) \in L^\infty(\Omega) \cap C(\bar{\Omega}),$$

$$u(x, k) = 0, \quad x \in \text{fr } \Omega, \quad u(x, k) = \exp(ikx) + \varphi(x, k),$$

$$\varphi(x, k) = O(|x|^{(1-N)/2}), \quad \left( \frac{\partial}{\partial |x|} - i\sqrt{\lambda} \right) \varphi(x, k) = o(|x|^{(1-N)/2}), \quad |x| \rightarrow \infty.$$

The existence and the properties needed by us of the functions  $u(x, k)$  have been proved, for example, in the work (1).

We also note the following very simply proved lemma:

**Lemma 1.** For  $x \in \Omega$  and  $\beta > N/2$ , the inequality

$$\int (1 + k^2)^{-\beta} |u(x, k)|^2 dk \leq (\pi)^{N/2} \Gamma(\beta)^{-1} \Gamma\left(\beta - \frac{N}{2}\right).$$

3. Let  $F(x, t) \in L^2(\Omega)$  for all  $t \geq 0$ . Put

$$\widehat{F}(k, t) = \lim_{R \rightarrow \infty} \int_{|x| \leq R} u(x, k) F(x, t) dx. \quad (2)$$

(The existence of the limit (2) in the metric of  $L^2$  was proved in (1).)

**Lemma 2.** If  $F(x, t)$  is continuous in  $t$  in the metric of  $L^2(\Omega)$ , then the solution  $v(x, t)$  of problem (1) exists, and its energy  $E(t)$  can be calculated by the formula

$$E(t) = (2\pi)^{-N} \int \left[ \left| \int_0^t \widehat{F}(k, \tau) \sin |k|(t - \tau) d\tau \right|^2 + \left| \int_0^t \widehat{F}(k, \tau) \cos |k|(t - \tau) d\tau \right|^2 \right] dk. \quad (3)$$

**Proof.** By the symbol  $\langle f, g \rangle$  we denote the scalar product in  $L^2(\Omega)$ . Let us note that it is sufficient to prove (3) for functions  $F(x, t)$  from  $C_0^\infty(\Omega)$  that are continuous in  $t$  (since they are dense in  $L^2(\Omega)$ ). For such functions  $F(x, t)$  in (1) the following equality is valid:

$$E(t) = \langle v_t, v_t \rangle + \langle \nabla_x v, \nabla_x v \rangle = \langle v_t, v_t \rangle - \langle \nabla_x^2 v, v \rangle = \langle v_t, v_t \rangle + \langle F(, t) - v_{tt}, v \rangle. \quad (4)$$

But

$$v(x, t) = (2\pi)^{-N} \int u^*(x, k) \left[ \int_0^t |k|^{-1} \sin |k|(t - \tau) \widehat{F}(k, \tau) d\tau \right] dk. \quad (5)$$

By virtue of Lemma (1), the derivatives with respect to  $t$  of the function  $v(x, t)$  may be computed by differentiating (5) under the integral sign; substituting them into (4) and using Parseval's equality

$$\langle f, g \rangle = (2\pi)^{-N} \int \hat{f}^*(k) \hat{g}(k) dk,$$

we obtain (3).

4. Let  $\omega_n$ ,  $n = 1, \dots$ , be real numbers,  $\omega_n \neq 0$ ,

$$\alpha_n(t, r) = \int_0^t \exp(i\omega_n \tau) \sin r(t - \tau) d\tau,$$

$$\beta_n(t, r) = \int_0^t \exp(i\omega_n \tau) \cos r(t - \tau) d\tau,$$

$$\gamma_{nm}(t, r) = \alpha_n(t, r) \alpha_m^*(t, r) + \beta_n(t, r) \beta_m^*(t, r).$$

**Lemma 3.** 1) If  $\omega_n \neq \omega_m$  and  $\varphi(r) \in L^1(0, \infty)$ , then

$$\lim_{t \rightarrow \infty} t^{-1} \int_0^\infty \gamma_{nm}(t, r) \varphi(r) dr = 0.$$

2) If  $|\omega_n|$  is a Lebesgue point of the function  $\varphi(r) \in L^1(0, \infty)$ , then

$$\lim_{t \rightarrow \infty} t^{-1} \int_0^\infty \gamma_{nn}(t, r) \varphi(r) dr = \pi \varphi(|\omega_n|).$$

Let us also note that if  $f(x) \in L^2(\Omega)$ , then for almost all  $r > 0$  the function

$$\theta(f)(r) = \int_{|n|=1} |\hat{f}(nr)|^2 r^{N-1} dn$$

is defined, and

$$\int_0^\infty \theta(f)(r) dr = \int |\hat{f}(k)|^2 dk = (2\pi)^N \int |f(x)|^2 dx.$$

5. We now consider the behavior of the energy  $E(t)$  in several typical cases. Let  $W(t) = E(t)/t$ . Put in (1)

$$F(x, t) = \sum_{n=1}^M f_n(x) \exp(i\omega_n t). \quad (6)$$

**Theorem 1.** If: 1) each of the functions  $f_n(x) \in L^2(\Omega)$ ; 2)  $\forall n : \omega_n \neq 0$  and  $|\omega_n|$  is a Lebesgue point of the function  $\theta(f)(r)$ , then

$$\lim_{t \rightarrow \infty} W(t) = (2\pi)^{-N} \pi \sum_{n=1}^M \theta(f_n)(|\omega_n|).$$

Let

$$F(x, t) = f(x) \sum_{n=1}^{\infty} a_n \exp(i\omega_n t). \quad (7)$$

**Theorem 2.** If  $f(x) \in L^1 \cap L^2$ , the series  $\sum a_n$  converges absolutely, and at least one of the following two conditions is fulfilled: 1) all numbers  $\omega_n$  satisfy the inequality

$$0 < a < |\omega_n| < b < \infty;$$

2) there exists a constant  $C < \infty$ , independent of  $x$  and  $k$ , such that

$$|u(x, k)| < C, \quad x \in \text{supp } f(x), \quad k \in R_N,$$

then

$$\lim_{t \rightarrow \infty} W(t) = (2\pi)^{-N} \pi \sum_{n=1}^{\infty} |a_n|^2 \theta(f)(|\omega_n|).$$

Theorems 1 and 2 are proved in the same way: it suffices to substitute (6) and (7) into (3) and use Lemma 3.

The physical meaning of the quantity  $\theta(f)(\omega)$  is explained by the following

**Lemma 4.** If  $F(x, t) = f(x) \exp(i\omega t)$ ,  $f(x) \in L^2(\Omega)$ , and the support of  $f(x)$  is compact and lies in  $\Omega$ , then

$$\lim_{t \rightarrow \infty} (E(t + \Delta t) - E(t)) = (2\pi)^{-N} \pi \theta(f)(|\omega|) \Delta t.$$

Theorems 1 and 2 may be regarded as assertions on the existence of an average limiting power for an almost periodic source of oscillations, and in this they are analogous to the well-known limiting-amplitude principle <sup>(2)</sup>.

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Moscow State University  
named after M. V. Lomonosov

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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