# ON AN INTEGRAL EQUATION IN THE THEORY OF GENERALIZED ANALYTIC FUNCTIONS IN THE SINGULAR CASE

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## Abstract

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**MATHEMATICS**

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# ON AN INTEGRAL EQUATION IN THE THEORY OF GENERALIZED ANALYTIC FUNCTIONS IN THE SINGULAR CASE

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As is known ($^1$), the investigation of a system of first-order differential equations in the plane, called the generalized Cauchy—Riemann system, is based on two-dimensional integral equations. If the coefficients of the system have first-order singularities, then we arrive at special integral equations. Some results for them were obtained in ($^2$); however, a complete theory has not yet been developed.

In the present paper we study the equation

$$W(z)=-\frac{\lambda}{\pi}\iint_{|\zeta|\leqslant 1}\frac{\overline{W(\zeta)}\,ds_\zeta}{\overline{\zeta}\,(\zeta-z)}+f(z),\qquad |z|\leqslant 1,\tag{1}$$

where $\lambda$ is an arbitrary complex number.

Putting $\zeta=\sigma z$, $\zeta=\rho e^{i\theta}$, $z=re^{i\varphi}$, $\sigma=\tau e^{i\alpha}$, we obtain

$$W(z)=-\frac{\lambda}{\pi}\iint_{\tau r<1}\frac{\overline{W(\sigma z)}}{\overline{\sigma}(\sigma-1)}\,ds_\sigma+f(z).\tag{2}$$

The method which we shall use resembles the method of separation of variables from the theory of differential equations. We shall seek the solution $W(z)=W(r,\varphi)$ in the form of a Fourier series with respect to the polar angle

$$W(r,\varphi)=\sum_{k=-\infty}^{\infty}W_k(r)e^{ik\varphi},\qquad W_k(r)=\frac{1}{2\pi}\int_0^{2\pi}W(r,\varphi)e^{-ik\varphi}\,d\varphi.\tag{3}$$

Analogous notation is used in expanding $f(z)=f(r,\varphi)$. Multiplying (2) by $\frac{1}{2\pi}e^{-ik\varphi}$ and integrating, we obtain

$$W_k(r)=-\frac{\lambda}{\pi}\iint_{\tau r<1}\frac{e^{ik\alpha}\overline{W_k(\tau r)}}{\overline{\sigma}(\sigma-1)}\,ds_\sigma+f_k(r)$$

and hence

$$W_k(r) = -\frac{\lambda}{\pi i}\int_0^1 \frac{\overline{W_{-k}(\rho)}}{\rho}\,d\rho \int_{|t|=1} \frac{t^k\,dt}{t - r/\rho} + f_k(r).$$

Using the residue theorem, we obtain

$$W_k(r) = -2\lambda \int_0^1 \frac{1}{\rho}\left(\frac{r}{\rho}\right)^k \overline{W_{-k}(\rho)}\,d\rho + f_k(r), \qquad k = 0, 1, 2, \ldots$$

$$W_{-k}(r) = 2\lambda \int_0^r \frac{1}{\rho}\left(\frac{\rho}{r}\right)^k \overline{W_k(\rho)}\,d\rho + f_{-k}(r), \qquad k = 1, 2, \ldots \tag{4}$$

Thus, in order to find each of the pairs of Fourier coefficients $W_k, W_{-k}$, we have a system of two integral equations. If the first line is substituted into the second, we arrive at a single equation

$$W_{-k}(r) = 4|\lambda|^2 \int_0^1 \mathcal{G}_k(r,\tau)\overline{W_{-k}(\tau)}\,d\tau + g_{-k}(r), \qquad k = 1, 2, \ldots, \tag{5}$$

where

$$\mathcal{G}_k(r,\tau) = \begin{cases} \dfrac{1}{2k}\dfrac{1}{\tau}\left(\dfrac{\tau}{r}\right)^k, & 0 \leqslant \tau \leqslant r, \\ \dfrac{1}{2k}\dfrac{1}{\tau}\left(\dfrac{r}{\tau}\right)^k, & r \leqslant \tau \leqslant 1; \end{cases} \tag{6}$$

$$g_{-k}(r) = f_{-k}(r) + 2\lambda \int_0^r \frac{1}{\rho}\left(\frac{\rho}{r}\right)^k \overline{f_k(\rho)}\,d\rho. \tag{7}$$

The integral equation (5) belongs precisely to the type of equations with kernels homogeneous of degree $-1$ that were studied in (3). Applying the results of (3), one can not only clarify questions of solvability of the equation in various Banach spaces, but also, in view of the analytic simplicity of the kernel $\mathcal{G}_k$, write the solution in explicit form. The kernel (6) satisfies the summability condition with exponent $\beta$, $-1 < \beta < 1$, and its characteristic function, determining normal solvability, $G_k(x) \neq 0$, $-\infty < x < \infty$, and the index $\varkappa = -\operatorname{Ind} G_0(x)$, will be

$$G_k(x) = 1 + 4|\lambda|^2 H_k(x), \qquad H_k(x) = \int_0^\infty \mathcal{G}_k(1,U)U^{ix-\beta}\,dU.$$

After computations this leads to the conclusion: the normality condition for equations (8) is satisfied for all $k = 1, 2, \ldots$ and all values of the parameter

$|\lambda|$, and the index $\varkappa = 0$. Hence it follows that all equations (5) are uniquely solvable for arbitrary free terms $g_{-k}(r)$ and any value of the parameter $|\lambda|$. In this case $W_{-k}(r)$ and $g_{-k}(r)$ may be considered in any of the Banach spaces $M_\beta, C_\beta, L^p_{\beta-1/p}$ (${}^2,{}^3$).

Having found $W_{-k}(r)$, $k = 1, 2, ...$, we substitute their values into the first line of the system (4), whence we obtain $W_k(r)$, $k = 1, 2, ....$ To find $W_0(r)$ we have an equation which, after complex conjugation and substitution into the original one, takes the form

$$W_0(r) = 4|\lambda|^2 \int_r^1 \frac{1}{\tau} \ln \frac{\tau}{r} \, w_0(\tau)\, d\tau - 2\lambda \int_r^1 \frac{\overline{f_0(\rho)}}{\rho}\, d\rho. \tag{8}$$

Again the kernel is homogeneous of degree $-1$; the summability condition is fulfilled for $\beta > 0$. Therefore, for the whole system (4) we shall take $0 < \beta < 1$.

The normality condition for equation (8) is violated when $|\lambda| = \beta/2$, while the index $\varkappa_0 = 0$ when $|\lambda| < \beta/2$ and $\varkappa_0 = -1$ when $|\lambda| > \beta/2$. When $|\lambda| < \beta/2$, the solution of (8) always exists and is unique, while when $|\lambda| > \beta/2$, for existence it is necessary and sufficient that the condition of orthogonality of the free–

term to a solution of the transposed homogeneous equation, which is $r^{2|\lambda|-1}$. Substituting the value of $f_0(r)$, the condition can be written in the form

$$\iint_{|z|\leqslant 1} |z|^{2|\lambda|-2} f(z)\, ds_z = 0. \tag{R}$$

We return to the original equation (1), assuming $f(z)$, $W(z) \in M_\beta$, $0 < \beta < 1$, and putting

$$\|f\| = \sup_{\substack{0<r\leqslant 1\\ 0<\varphi<2\pi}} r^\beta |f(r,\varphi)|.$$

Then from (3) it immediately follows that $W_k(r) \in M_\beta$, and similarly $f_k(r) \in M_\beta$.

**Theorem.** The homogeneous integral equation (1) has no solutions, other than the zero solution, in the class $M_\beta$, $0 < \beta < 1$, for any complex $\lambda$.

Let $f(z) \in M_\beta$, $0 < \beta < 1$, and be such that the series converges

$$\sum_{k=-\infty}^{\infty} \|f_k(r)\|_{M_\beta} < +\infty. \tag{9}$$

Then, for $|\lambda| < \beta/2$, the nonhomogeneous equation is uniquely solvable for every $f(z)$. If, however, $|\lambda| > \beta/2$, then for a solution to exist it is necessary and sufficient that the free term $f(z)$ satisfy condition (R).

**Proof.** From (5)–(7) we have (we omit the indication of the space $M_\beta$ in the notation of the norm)

$$\|g_{-k}\| \leqslant 2\,\frac{|\lambda|}{k-\beta}\,\|f_k\| + \|f_{-k}\|,$$

$$\|W_{-k}\| \leqslant \frac{4|\lambda|^2}{k^2-\beta^2}\,\|W_{-k}\| + \|g_{-k}\|.$$

Take $k = k_0$ so large that $4|\lambda|^2/(k_0^2-\beta^2) \leqslant 1/2$; then

$$2|\lambda|/(k_0+\beta) \leqslant 2|\lambda|/(k_0-\beta) \leqslant 1,$$

so that, beginning with this $k = k_0$, we have

$$\|g_{-k}\| \leqslant \|f_k\| + \|f_{-k}\|,$$

$$\|W_{-k}\| \leqslant \tfrac{1}{2}\|W_{-k}\| + \|g_{-k}\|, \quad \text{or} \quad \|W_{-k}\| \leqslant 2\|g_{-k}\|,$$

$$|k| \geqslant k_0,$$

whence

$$\|W_{-k}\| \leqslant 2(\|f_k\| + \|f_{-k}\|),$$

and further, from the first equation of system (7),

$$\|W_k\| \leqslant \|W_{-k}\| + \|f_k\| \leqslant 3(\|f_k\| + \|f_{-k}\|).$$

Thus,

$$\|W_k\| < 3(\|f_k\| + \|f_{-k}\|), \qquad |k| \geqslant k_0,$$

thanks to which condition (9) ensures the convergence of the series

$$\sum_{k=-\infty}^{\infty} \|W_k(r)\| = \sum_{k=-\infty}^{\infty} \sup_{0<r\leqslant 1} r^\beta |W_k(r,\varphi)|,$$

from which, in turn, follows the absolute and uniform convergence with respect to $r, \varphi$ of the Fourier series (3).

The series (3) will indeed give a solution of equation (1), since all operations performed when substituting the series into (1) will be justified. The theorem is proved.

The restriction (9) has been imposed because of the method, namely in order to ensure convergence of the series (1). The convergence condition for the series (9), apparently, can be considerably weakened. It is ensured, for example, by the usual-

smoothness conditions of the type $r^\beta f(r,\varphi) \in C^2$, etc. Let us also note that by the substitution $W = W_1 + f$ we could reduce (1) to an equation with a free term which is differentiable with respect to $\bar{z}$ for $|z| > 0$, and thereby the convergence of the Fourier series for $r > 0$ would be ensured. Requirement (9) has to be imposed in view of the presence of the singular point $z = 0$.

**Remark 1.** Analogous results could be obtained in the spaces $C_\beta$, $L^p_\beta$, etc. Moreover, it is clear that Theorem 1 is valid for a disk of arbitrary radius. In the case when the domain of integration is the whole plane, Theorem 1 is simplified in the sense that the solvability condition is absent.

**Remark 2.** Consider the equation

$$W(z) = -\frac{1}{\pi} \iint_D \frac{B(\zeta)\overline{W(\zeta)}}{\zeta(\zeta - z)}\, d\xi_\zeta + f(z), \tag{10}$$

where $B(\zeta) \in M(D)$ and is continuous at the point $\zeta = 0$, the domain $D$ is bounded and $0 \in D$. By the usual subtraction procedure we isolate the simplest singular part, and the second term will be a completely continuous operator. It then follows from the general theorems of operator theory that for equation (10) in $M_\beta$ the Fredholm theorems are valid with index $\varkappa = 0$ when $|B(0)| < \beta/2$ and $\varkappa = -1$ when $|B(0)| > \beta/2$.

**Remark 3.** Let $f(z)$ be bounded. Then $f(z) \in M_\beta$ with arbitrarily small $\beta > 0$, and, applying the theorem, we obtain that for every $\lambda$ one solvability condition (R) is required, and the index is $\varkappa = -1$. However, the solution $W(z)$ itself, generally speaking, will not be bounded at the point $z = 0$.

Suppose the condition

$$f_0(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r,\varphi)\, d\varphi = 0 \tag{11}$$

is satisfied.

Then, first, (R) is fulfilled, and second, since equation (8) will be homogeneous, $W_0(r) \equiv 0$. All the remaining equations (5) and (4) could be considered with $\beta = 0$, i.e., if $f_k(r) \in M$, then also $W_k(r) \in M$.

Thus, if $f(z) \in M$ and condition (11) is satisfied, then a solution of (1) exists, is unique, and is bounded for every $\lambda$.

The differential equation

$$\frac{\partial W}{\partial \bar{z}} = \frac{\lambda}{z}\overline{W}$$

is reduced to the integral equation (1), where $f(z)$ will be a holomorphic function of $z$. Then (11) is equivalent to the requirement $f(0) = 0$. A one-to-one correspondence is established between the families of generalized analytic functions $W(z)$ and analytic functions $f(z)$, $f(0) = 0$.

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*Note: Figure translations are in progress. See original paper for figures.*