Reports of the Academy of Sciences of the USSR

1970

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1970. Vol. 194, No. 4

Aerodynamics

A. F. Sidorov

On Exact Solutions of the Equations of Gas Dynamics of the Triple-Wave Type

(Presented by Academician L. I. Sedov on 18 III 1970)

The overdetermined system of equations describing potential nonstationary triple waves in a polytropic gas ([1]) has the form

$$\Gamma_j = \sum_{i,k} A_{ik}(F) L_{ik}^{(j)}(F, \Pi) = 0, \qquad i, k, j = 1, 2, 3; \tag{1}$$

$$L_{ik}^{(1)} = (-1)^{i+k} \begin{vmatrix} \varkappa F_{mp} + \delta_{mp} & \varkappa F_{np} + \delta_{np} \\ \varkappa F_{mq} + \delta_{mq} & \varkappa F_{nq} + \delta_{nq} \end{vmatrix},$$

$$L_{ik}^{(2)} = (-1)^{i+k} \left\{ \begin{vmatrix} \varkappa F_{mp} + \delta_{mp} & \varkappa F_{np} + \delta_{np} \\ \varkappa \Pi_{mq} + \delta_{mq} & \varkappa \Pi_{nq} + \delta_{nq} \end{vmatrix} + \begin{vmatrix} \varkappa \Pi_{mp} + \delta_{mp} & \varkappa \Pi_{np} + \delta_{np} \\ \varkappa F_{mq} + \delta_{mq} & \varkappa F_{nq} + \delta_{nq} \end{vmatrix} \right\},$$

$$L_{ik}^{(3)} = (-1)^{i+k} \begin{vmatrix} \varkappa \Pi_{mp} + \delta_{mp} & \varkappa \Pi_{np} + \delta_{np} \\ \varkappa \Pi_{mq} + \delta_{mq} & \varkappa \Pi_{nq} + \delta_{nq} \end{vmatrix},$$

$$m, n \neq k,\ m < n; \qquad p, q \neq i,\ p < q;$$

$$A_{ik} = \delta_{ik} F - \varkappa^2 F_{iF} k, \tag{2}$$

$$F_i = \partial F / \partial u_i, \qquad F_{ik} = \partial^2 F / \partial u_i \partial u_k, \qquad \Pi_i = \partial \Pi / \partial u_i, \qquad \Pi_{ik} = \partial^2 \Pi / \partial u_i \partial u_k.$$

The system of three second-order equations (1) is satisfied by the unknown functions $F(u_1, u_2, u_3) = c^2$ (c is the speed of sound) and $\Pi(u_1, u_2, u_3)$, the placement function; δ_{ik} is the Kronecker symbol, $\varkappa = 1/(\gamma - 1)$; γ is the adiabatic exponent in the equation of state $p = a^2\rho^\gamma$; p is pressure; ρ is density; $a^2 =$ const, u_i are the components of the velocity vector $\mathbf{u}$. In the case of a triple wave, to a flow region in four-dimensional physical space x_1, x_2, x_3, t (x_i are spatial coordinates, t is time) there corresponds, in the space of the unknown functions u_1, u_2, u_3, c, a manifold of dimension 3. We shall regard u_1, u_2 and u_3 as functionally independent. After F and Π have been found from (1), the flow in physical space is reconstructed by the formulas

$$x_i = \varkappa\Pi_i + u_i + t(\varkappa F_i + u_i), \qquad i = 1, 2, 3. \tag{3}$$

In ([2]), for the case of an isothermal gas, a class of exact solutions of system (1) was found, depending on 3 functions of one argument. There, for a polytropic gas, a self-similar flow was also obtained, with the aid of which a solution was constructed for the problem of outflow into a vacuum along a certain dihedral angle. The question of the nonemptiness and the physical content of the class of non-self-similar triple waves for $\gamma \neq 1$ remained open. In the present note a family of exact solutions of the equations of hydrodynamics of the type of a non-self-similar triple wave is constructed for $2 > \gamma > 1$, depending on three arbitrary functions of one argument, and some of its applications and features are investigated.

1. We shall seek solutions of system (1) in the form

$$F = [a_0 + (\mathbf{a}\cdot\mathbf{u})]^2, \qquad |\mathbf{a}|^2 = 3/4\varkappa(\varkappa - 1), \qquad a_0 = \text{const}, \tag{1.1}$$

$$\Pi = \sum_{i=1}^{3} \left(-\frac{1}{2\chi} u_i^2 + T_i \left[(\alpha_i \cdot \mathbf{u}) \right] \right), \tag{1.2}$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\alpha_i = (\alpha_{1i}, \alpha_{2i}, \alpha_{3i})$ are some constant linearly independent vectors. For $T_i \equiv 0$ and F from (1.1), the corresponding self-similar solution was obtained in ([2]). For $j = 1$ in (1), the equation for F is independent and, by virtue of (1.1), is satisfied automatically. For the function Π from (1), there remains a system of two equations ($j = 2, 3$). Substituting Π from (1.2) into these two equations, we reduce them to the form

$$\sum_{i=1}^{3} \left(|\alpha_i|^2 - \frac{3-\gamma}{(\gamma-1)^2} |\mathbf{a}\times\alpha_i|^2 \right) T_i' \left[(\alpha_i\cdot\mathbf{u}) \right] = 0, \tag{1.3}$$

$$\sum_{i,k=1}^{3} \left(|\alpha_i\times\alpha_k|^2 - \frac{4}{(\gamma-1)^2} |\mathbf{a}\cdot(\alpha_i\times\alpha_k)|^2 \right) T_i'' \left[(\alpha_i\cdot\mathbf{u}) \right] T_k'' \left[(\alpha_k\cdot\mathbf{u}) \right] = 0,$$

where primes denote differentiation of the functions T_i with respect to their full arguments.

In order that the exact solution (1.1), (1.2) depend on 3 arbitrary functions T_1, T_2, and T_3, it is necessary and sufficient that the conditions

$$|\alpha_i \times \alpha_k|^2 - \frac{4}{(\gamma-1)^2} \left|\mathbf{a} \cdot (\alpha_i \times \alpha_k)\right|^2 = 0, \tag{1.4}$$

$$|\mathbf{a} \times \alpha_i|^2 = (\gamma-1)^2/(3-\gamma) \tag{1.5}$$

be satisfied (without loss of generality, one may set $|\alpha_i| = 1$).

From geometric considerations it is clear that the system of equations (1.4), (1.5), for a given $\mathbf{a}$ ($|\mathbf{a}|^2 = 3(\gamma-1)^2/4(2-\gamma)$), will be satisfied only when the α_i form equal angles with one another and are equally inclined to $\mathbf{a}$. Thus, for example, for $\mathbf{a} = (0, 0, a_3)$, one may take as α_i

$$\alpha_1 = (\sin\nu, 0, \cos\nu),$$

$$\alpha_2 = \left(-\frac{1}{2}\sin\nu, \frac{\sqrt{3}}{2}\sin\nu, \cos\nu\right), \tag{1.6}$$

$$\alpha_3 = \left(-\frac{1}{2}\sin\nu, -\frac{\sqrt{3}}{2}\sin\nu, \cos\nu\right),$$

where

$$\sin\nu = \frac{2}{\sqrt{3}} \frac{\sqrt{2-\gamma}}{\sqrt{3-\gamma}} \qquad (2 > \gamma > 1). \tag{1.7}$$

In this case the additional condition on the volume of the parallelepiped formed from the vectors $\mathbf{a}, \alpha_i, \alpha_k$, following from (1.4), is satisfied automatically.

Thus, for α_i defined by (1.6), formulas (1.1), (1.2) generate an exact solution of equations (1) with arbitrary T_i.

It is clear that if one sets $T_3 \equiv 0$, or $T_3 \equiv 0$ and $T_2 \equiv 0$, when the solutions (1.1), (1.2) depend respectively on two and on one arbitrary function, then in finding α_i there remains greater arbitrariness than in the case $T_i \equiv 0$.

Analogues of the solutions (1.1), (1.2) in the case of plane unsteady flows of the double-wave type will be flows determined by the formulas

$$x_1 = \left[u_1 + \frac{\gamma - 1}{2}(a_0 + a_1 u_1 + a_2 u_2) a_1\right] t + F_1'(u_1 + \beta_1 u_2) + F_2'(u_1 + \beta_2 u_2), \quad (1.8)$$

$$x_2 = \left[u_2 + \frac{\gamma - 1}{2}(a_0 + a_1 u_1 + a_2 u_2) a_2\right] t + \beta_1 F_2'(u_1 + \beta_1 u_2) + \beta_2 F_2'(u_1 + \beta_2 u_2),$$

where

$$a_1^2 + a_2^2 = 4/(3 - \gamma), \qquad 3 > \gamma > 1,$$

$$\beta_1 = \left(-a_1 a_2 + \sqrt{a_1^2 + a_2^2 - 1}\right)/(1 - a_1^2),$$

$$\beta_2 = \left(-a_1 a_2 - \sqrt{a_1^2 + a_2^2 - 1}\right)/(1 - a_1^2),$$

and the functions F_1' and F_2' are arbitrary.

For $F_1' \equiv F_2' \equiv 0$ such a solution was considered in ([3]). Of course, each time arbitrary functions are chosen it is necessary to check the solvability conditions of equations (3) and (1.8) for u_i.

§ 2. Solutions (1.1), (1.2) make it possible to construct flow fields in certain problems on the interaction of three plane one-dimensional Riemann waves. One such solution for an isothermal gas was investigated in ([2]). In the planar case, problems on the interaction of two Riemann waves were considered in ([3–6]).

Let, at the initial instant of time $t = 0$, a polytropic homogeneous gas be at rest inside a certain trihedral infinite angle formed by the planes P_i, orthogonal respectively to the vectors $\alpha_2 \times \alpha_3$, $\alpha_1 \times \alpha_3$, $\alpha_1 \times \alpha_2$. The planes P_i begin to move in the gas parallel to themselves with velocities $V_i(t)$. It is clear that far from the vertex and the edges of the angle there arise plane Riemann flows; far from the vertex near the edges, regions of interaction of two Riemann waves, where one may try to construct the flow in the class of plane double waves; and, finally, at the vertex of the angle in the region of interaction of the double waves, one may seek a solution in the class of triple waves. True, in this case ([6]) it is not always possible, even in the class of double waves, to construct the flow as a whole—one has to introduce certain curvilinear moving walls. However, part of the interaction region in the class of double waves can always be constructed.

For the case under consideration, setting in (1.1), (1.2), for fixed i, $(\alpha_i \cdot u) \equiv 0$, we obtain a flow of the type of a plane double wave (instead of (3), two equations remain, obtained by composing the corresponding linear combination); setting two such relations equal to zero at once, we obtain plane Riemann waves. Each time, in accordance with the theorem ([7]) on the matching of flows of different ranks, planes of the type $(\alpha_i{\cdot}u) = 0$ or straight lines $((\alpha_i{\cdot}u) = 0,\ (\alpha_k{\cdot}u) = 0,\ i \neq k)$ in the velocity hodograph space u_1, u_2, u_3 will be characteristic manifolds, respectively, for the equations of triple and double waves. Thus, in the case where the potentiality of the flow is preserved, one can, with the aid of (1.1), (1.2), construct a solution in some region of interaction of three Riemann waves (the functions T_i are determined by the prescribed $V_i(t)$).

The possibility of reconstructing flows in the physical space x_1, x_2, x_3, t depends on the property of the Jacobian $J = \partial(x_1, x_2, x_3)/\partial(u_1, u_2, u_3)$, which must not vanish in the domain of definition of the flow in the space u_1, u_2, u_3, t. If J vanishes, limiting manifolds appear in the flow and a gradient catastrophe occurs. It is of interest to ascertain the features of the appearance of limiting manifolds and the destruction of potential traveling waves for the class of flows (1.1), (1.2). Such destruction will inevitably occur at a finite instant of time $t = t^*$, if the planes P_i move into the gas, forming compression waves.

Writing (3) in the form

$$x_i = T_1'\alpha_{i1} + T_2'\alpha_{i2} + T_3'\alpha_{i3} + t(2\chi\sqrt{F} + u_i), \tag{2.1}$$

we compute J. It turns out that J can be represented in the form

$$J = \frac{\gamma+1}{2(2-\gamma)}\left(t + 2\frac{2-\gamma}{3-\gamma}T_1''\right)\left(t + 2\frac{2-\gamma}{3-\gamma}T_2''\right)\left(t + 2\frac{2-\gamma}{3-\gamma}T_3''\right). \tag{2.2}$$

An analogous representation is valid for the Jacobians in the case of plane double waves, when two brackets remain in (2.2), and in the case of Riemann waves.

Directly from the form of J we obtain the theorem: the phenomenon of a gradient catastrophe in the class of flows (1.1), (1.2) occurs simultaneously along a certain surface in the space x_1, x_2, x_3, both in the region of the triple wave and in the two regions of double waves adjacent to it, and in one of the Riemann waves.

Thus, in the class of flows (1.1), (1.2), no local destruction of the potential motion can occur.

It is clear that the results and the form of the exact solutions of (1.1), (1.2) are also preserved for the equation of state of the gas $p = a^2(\rho^\gamma - \rho_0^\gamma)$, $\rho_0 = \text{const}$.

Sverdlovsk Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
10 XI 1969

References

[1] A. F. Sidorov, PMM, **23**, issue 5 (1959).
[2] A. F. Sidorov, PMM, **28**, issue 6 (1964).
[3] V. A. Suchkov, PMM, **27**, issue 4 (1963).
[4] Yu. A. Pogodin, V. A. Suchkov, N. P. Yanenko, PMM, **22**, issue 2 (1958).
[5] E. V. Ermolin, A. F. Sidorov, PMM, **30**, issue 2 (1966).
[6] E. V. Ermolin, L. N. Rubina, A. F. Sidorov, PMM, **32**, issue 5 (1968).
[7] A. F. Sidorov, O. B. Khairullina, PMM, **33**, issue 1 (1969).

Note: Figure translations are in progress. See original paper for figures.