EXTENSION OF MAPPINGS INTO SPHERES AND P. S. ALEKSANDROV' S PROBLEM ON BICOMPACT COMPACTIFICATIONS\

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Abstract

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EXTENSION OF MAPPINGS INTO SPHERES AND P. S. ALEKSANDROV' S PROBLEM ON BICOMPACT COMPACTIFICATIONS*

(Presented by Academician P. S. Aleksandrov on 12 III 1970)

In the present note two of the following problems of Yu. M. Smirnov, connected with a theorem of W. Sierpiński ([1]) and with the problem of P. S. Aleksandrov mentioned in the title ([2]), are solved affirmatively.

Yu. M. Smirnov observed that W. Sierpiński' s theorem on the indecomposability of continua into a countable union of pairwise disjoint bicompacts can be modified in the following way:

If a bicompactum X is a countable union of pairwise disjoint bicompacts, among which there is the zero-dimensional sphere S^0, then X retracts onto S^0.

In ([2]), in connection with the construction of an example solving the problem of P. S. Aleksandrov formulated below, arguments are given which, almost without change, are suitable for the proof of one natural generalization of this assertion to the one-dimensional case under the assumption that the bicompactum X is metrizable.

In 1967 in Warsaw Yu. M. Smirnov posed the following question:

Will the set X_0 be a retract of the bicompactum $X = \bigcup_{i=0}^{\infty} X_i$ in the case when X_0 is an n-dimensional sphere, $\dim(X_i \cap X_j) \leq n-1$ for $i \neq j$, and the sets X_i are closed? An affirmative solution of this problem is given by our Theorem 1**. In ([2]) serious attention was first drawn to the connection of dimension theory with the problem of P. S. Aleksandrov that interests us and that for a long time had not yielded to solution:

Do there exist bicompact compactifications of connected, locally connected complete metric spaces with a countable base that are countable unions of compacta?

There (([2]), Theorem 2) one sufficient condition for bicompact compactifiability was given, formulated in terms of connectedness and local connectedness in dimension n***. Therefore there also arose a second question, on the construction of examples of spaces connected and locally connected in dimension n satisfying the conditions of P. S. Aleksandrov but having no bicompact compactifications

(Theorem 5). Moreover, from Theorem 1 there are derived: a sufficient condition for the absence of bicompact compactifications (Theorem 2) and a sufficient condition for extendability of mappings into a sphere (Theorems 3 and 4).

Theorem 1. *If a bicompactum X is a countable union of bicompacts whose pairwise intersections**** are at most $n-1$-dimensional, then every mapping of each of them into the n-dimensional sphere is extendable to all of X*****.*

For the proof we need the following three lemmas. To formulate the first, it is convenient to regard the sphere S^n as the boundary of the $n+1$-dimensional cube $[-1,1]^{n+1}$, lying in E^{n+1}, and to denote its faces as follows:

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* A compactification of a space is understood to mean any one-to-one and continuous image of it.

** A special case of this problem is solved by V. Goldshtein in ([3]). The proof given there is of a character different from the present work and, moreover, contains an error.

*** For the definitions of these notions, see below.

**** For different summands.

***** We consider only continuous mappings.

$$S^n_{\varepsilon i} = \{x \in [-1,1]^{n+1} \mid x_i = \varepsilon\}, \quad i = 1, 2, \dots, n+1,$$

where x_i are the coordinates of the point x, and $\varepsilon = \pm 1$.

Lemma 1. A continuous mapping $f : A \to S^n$ of a closed set A, lying in a normal space X, is continuously extendable over all of X if and only if in X there exist such closed partitions C_i* between the sets $f^{-1}S^n_i$ and $f^{-1}S^n_{-i}$ that

$$\bigcap_{i=1}^{n+1} C_i = \emptyset^{**}.$$

The following two lemmas require the notion of normal adjoining, introduced for an entirely different reason by Yu. M. Smirnov ([4]):

A subset N of a space X **normally adjoins** its complement $M = X \setminus N$ if any two disjoint closed subsets of M have disjoint open neighborhoods in X.

Obviously, every subset of a hereditarily normal space normally adjoins its complement, and also every open subset of a normal space normally adjoins its complement. The following is proved quite simply.

Lemma 2. Every set of type G_δ of a normal space normally adjoins its complement.

Lemma 3, it seems to us, is also of independent interest:

Lemma 3. For any set M of dimension $\dim M \leq n$, with complement $X \setminus M$ normally adjoining it in a normal space X, every system of $n+1$ pairs of closed subsets $A_{\varepsilon i}$ in X, where $\varepsilon = \pm 1$, $i = 1, 2, \dots, n+1$, and $A_i \cap A_{-i} = \emptyset$ for all i, can be separated by closed partitions C_i in X between A_i and A_{-i} so that

$$M \cap \bigcap_{i=1}^{n+1} C_i = \emptyset.$$

We shall now give consequences of Theorem 1.

Theorem 2. A topological space X which is a countable union of bicompacts, the pairwise intersections of which are at most $(n-1)$-dimensional, has no bicompact compactifications if there exists at least one such mapping of one of the summand bicompacts into the n-dimensional sphere that is not extendable over all of X***.

The natural desire to get rid of the bicompactness of the space X and of its summands in Theorems 1 and 2 is unrealizable without some additional assumptions: there exists an example of a planar, one-dimensional, connected, locally bicompact set K, which is a countable union of pairwise disjoint connected closed subsets. Combining any two of its summands K_0 and K_1 into one and putting $f(K_0) = 0$, and $f(K_1) = 1$, we obtain a mapping into the zero-dimensional sphere, not extendable over the whole set K (see ([5]), p. 183). On the other hand, the space K is locally bicompact and, consequently, has bicompact compactifications.

Therefore the following theorems may be of interest:

Theorem 3. Let a completely regular space X be a countable union of closed sets, the pairwise intersections of which are at most $(n-1)$-dimensional and of which no more than one, say X_0, is non-bicompact, but has locally bicompact complement $X \setminus X_0$; then any mapping $f : X_0 \to S^n$, extendable over some neighborhood QK_0, whose complement is bicompact, is extendable over all of X****.

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* A set C is called a partition between the sets A and B if $X \setminus C = OA \cup OB$, where OA and OB are open in X and $OA \cap OB = \emptyset$.

** It follows from this that the equivalence of Lemmas 3 and 3′ asserted in ([2]) is in fact valid.

*** Consequence 1 of Lemmas 3′ and 4′ of work ([2]) follows from this.

**** Or, in other words, if X_0 contains the set of those points at which the space is not locally bicompact and $f : X_0 \to S^n$ is extendable over the growth of some bicompact extension of the space X, then f extends over this entire extension and, a fortiori, over all of X.

The theorem ceases to be true if at least two of the summands are not bicompact. Completely abandoning the requirement of bicompactness of the summands permits only the following

Theorem 4. *Let a locally bicompact space X be a countable union of closed sets whose pairwise intersections are at most $n-1$-dimensional, where $n \geqslant 1$; then every mapping of each summand into the n-dimensional sphere, extendable to the nonproper point of the minimal bicompact extension ωX, is extendable also to all of ωX, and hence also to all of X.*

The theorem does not cease to be true if we consider a bicompact extension cX of a locally bicompact space X whose remainder is finite.

Let us proceed to the construction of an example. We shall use the following definitions of connectedness and local connectedness in dimension n:

A space X will be called **connected in dimension** n if, for every bicompactum B of dimension $\dim B \leqslant n$, any two mappings of the bicompactum B into X are homotopic; a space X will be called **locally connected in dimension** n if for every point x and for every neighborhood Ox of it there exists such a neighborhood Ux of it that any two mappings of an arbitrary bicompactum B of dimension $\dim B \leqslant n$ carrying B into Ux are homotopic in Ox.

Theorem 5. *The space A_n defined below is an $n+1$-dimensional complete metric space with a countable base, connected and locally connected in dimension $n-1$, decomposable into a countable union of compacta, but having no bicompactifications.*

Definition of the space A_n. Let Q^{n+1} be the unit ball of the space E^{n+1}, and let S^n be its boundary. Consider a sequence of simplicial subdivisions P_i, $i = 1, 2, 3, \dots$, of the sphere S^n, each of which is a simplicial subdivision of the preceding one, and let the mesh of the subdivision P_i be less than $1/2^{i+1}$. Denote by L_i the $n-1$-dimensional skeleton of the subdivision P_i, and finally consider in E^{n+2} the set

$$A_n = (S^n \times 0) \cup \bigcup_{k=1}^{\infty} \left(Q^{n+1} \times \frac{1}{k}\right) \cup \bigcup_{k=1}^{\infty} \left(L_k \times \left[0, \frac{1}{k}\right]\right).$$

Lemma 4. *The space*

$$A'_n = (S^n \times 0) \cup \bigcup_{k=1}^{\infty} \left(Q^{n+1} \times \frac{1}{k}\right)$$

cannot be retracted onto $S^n \times 0$.

From this and from Theorem 2 it follows that the space A_n has no bicompactifications. To prove that the basic properties of connectedness and local connectedness in dimension $n-1$ hold for us, we shall first consider in E^{n+1}

a sequence of spheres $S_i^n,\ i = 1, 2, \dots$, where all the spheres S_i^n are concentric with the sphere S^n, and the corresponding numbers $1 + 1/i$ are the lengths of their radii. Let H_i be that part of the cone over L_i with vertex at the center of the sphere S^n which lies in the closed layer between S^n and S_i^n.

Lemma 5. *The set*

$$B = S^n \cup \bigcup_{k=1}^{\infty} S_k^n \cup \bigcup_{k=1}^{\infty} H_k$$

is connected and locally connected in dimension $n - 1$.

We note that for $n = 1$ the example A_n and Theorem 5 pass respectively into example A and theorem A of the paper ([2]).

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Note: Figure translations are in progress. See original paper for figures.