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SEVERAL THEOREMS ON THE STRONG LAW OF LARGE NUMBERS AND THE LAW OF THE ITERATED LOGARITHM

MATHEMATICS

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Abstract

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MATHEMATICS

V. A. EGOROV

SEVERAL THEOREMS ON THE STRONG LAW OF LARGE NUMBERS AND THE LAW OF THE ITERATED LOGARITHM

(Presented by Academician Yu. V. Linnik on 4 I 1970)

1. In the paper of V. V. Petrov ([1]) the order of growth of sums of independent random variables was investigated under very general assumptions. In the present article the order of growth of sums of independent random variables is studied under certain assumptions on the smallness of the individual summands relative to the sequence of normalizing constants. In addition, the article obtains strengthenings of some theorems of ([1]), and also investigates the connection between the order of decrease of Lindeberg-type fractions and the law of the iterated logarithm.

As in ([1]), introduce the classes of functions Ψ_c and Ψ_d. A function f belongs to Ψ_c (respectively Ψ_d) if $f(x)$ is positive and nondecreasing in the region $x > x_0$ for some x_0, and the series

$$\sum_n \frac{1}{nf(n)}$$

converges (respectively diverges). In addition, introduce the set of functions F_c such that, for some positive k, the function $[f(x)]^k$ belongs to Ψ_c, and the set of functions F_d such that, for every $k > 0$, the function $[f(x)]^k$ belongs to Ψ_d.

It is easy to see that if the function $(\ln x)^{\delta(x)}$ is nondecreasing for $x > x_0$, then it belongs to the class Ψ_c if $\delta(x) > 1+\gamma$, to the class Ψ_d if $\delta(x) \leq 1$, to the class F_c if $\delta(x) > \gamma$, and to the class F_d if $\delta(x) \to 0$ $(x \to \infty)$. Here γ is an arbitrary positive number.

2. Let $\{X_n\}$ be a sequence of independent random variables, $EX_n = 0$, $EX_n^2 < \infty$ $(n = 1, 2, ...)$.

Introduce the notation

$$S_n = \sum_{k=1}^{n} X_k, \qquad DX_n = EX_n^2, \qquad B_n = \sum_{k=1}^{n} DX_k.$$

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Theorem 1. Let $B_n \to \infty$ and $f \in F_c$. If

$$\sum_n P\left(|X_n| > \varepsilon\sqrt{B_n f(B_n)}\right) < \infty \quad \text{for every } \varepsilon > 0, \tag{1}$$

then

$$S_n = o\left(\sqrt{B_n f(B_n)}\right) \quad \text{a.s.} \tag{2}$$

By the Borel-Cantelli lemma, condition (1) is necessary for relation (2) to hold.

Theorem 2. For every $f \in F_d$ there exists a sequence of independent random variables $\{X_n\}$ with mathematical expectations equal to zero and finite variances such that $B_n \to \infty$, condition (1) is fulfilled, and

$$\limsup \frac{|S_n|}{\sqrt{B_n f(B_n)}} = \infty \quad \text{a.s.} \tag{3}$$

Theorem 3. For every function $f \in \Psi_d$ there exists a sequence of independent random variables with mathematical expectations equal to zero and finite variances such that $B_n \to \infty$ and relation (3) holds.

Theorem 3 strengthens Theorems 2 and 3 of paper ([1]).

Analogous investigations can also be carried out in the case of the strengthened law of large numbers with the simplest normalization:

$$S_n/n \to 0 \quad \text{a.s.} \tag{4}$$

Theorem 4. *If* $\sum_n P(|X_n| > \varepsilon n) < \infty$ *for every* $\varepsilon > 0$ *and*

$$B_n = O(n^2/f(n)) \tag{5}$$

for some function f from F_c, then relation (4) holds.

On the other hand, for every function f from F_d such that $f(x) = o(x^{2-\alpha})$ as $x \to \infty$ for some $\alpha > 0$, there exists a sequence of independent random variables $\{X_n\}$ with finite variances and zero means, for which $B_n \to \infty$, $\sum_n P(|X_n| > \varepsilon n) < \infty$ for every $\varepsilon > 0$, but

$$\limsup S_n/n = \infty \quad \text{a.s.} \tag{6}$$

Theorem 5. *If for some function $f \in \Psi_c$ relation (5) is satisfied, then relation (4) holds. On the other hand, for every function f from Ψ_d such that $f(x) = o(x^{2-\alpha})$ as $x \to \infty$ for some $\alpha > 0$, there exists a sequence of independent random variables $\{X_n\}$ with finite variances and zero means for which relations (5) and (6) are satisfied.*

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Theorem 5 is a strengthening of Theorem 4 of paper ([1]).

3. Let $g(x)$ be an even continuous function, positive and strictly increasing in the region $x > 0$, with $g(x) \to \infty$ as $x \to \infty$.

Let the functions $g(x)$ satisfy one of the following two conditions:

(A) $x/g(x)$ is nondecreasing in the region $x > 0$.

(B) $x/g(x)$ and $g(x)/x^2$ are nonincreasing in the region $x > 0$.

Assume that $Eg(X_k) < \infty$ $(k = 1, 2, ...)$, and let

$$A_n = \sum_{k=1}^{n} Eg(X_k).$$

In the case when condition (B) is satisfied, we shall assume that $EX_k = 0$ $(k = 1, 2, ...)$.

Theorem 6*. *Let condition (A) or (B) be satisfied and $f \in F_c$. If, for every $\varepsilon > 0$, the series*

$$\sum_n P\left(|X_n| > \varepsilon g^{-1}(A_n f(A_n))\right) < \infty \tag{7}$$

and $A_n \to \infty$, then

$$S_n = o\left(g^{-1}(A_n f(A_n))\right) \quad \text{a.s.} \tag{8}$$

By virtue of the Borel-Cantelli lemma, relation (7) is necessary for the validity of (8).

Theorem 7. *Let $g(x)$ be an even continuous function, positive and strictly increasing in the region $x > 0$, with $g(2x)/g(x) > 1 + \delta$ for some $\delta > 0$.***

For every function $f \in F_d$ there exists a sequence of independent symmetric random variables $\{X_n\}$ with $Eg(X_n) < \infty$ such that $A_n \to \infty$, condition (7) is satisfied, but

$$\limsup \frac{|S_n|}{g^{-1}(A_n f(A_n))} = \infty \quad \text{a.s.} \tag{9}$$

* g^{-1} denotes the function inverse to g.

** In particular, these conditions are satisfied if condition (A) or (B) is satisfied.

Theorem 8. Let $g(x)$ be an even continuous function, positive and strictly increasing in the domain $x > 0$, with $g(2x)/g(x) < C$ for some positive constant

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C. In particular, these conditions will be satisfied if one of conditions (A) or (B) is satisfied.

For any function $f \in \Psi_d$ there exists a sequence of independent symmetric random variables $\{X_n\}$ with $Eg(X_n) < \infty$ $(n = 1, 2, ...)$ such that $A_n \to \infty$ and relation (9) is satisfied.

Theorems 9, 10, 11 are consequences of Theorems 6, 7, 8.

Theorem 9. Let, for some positive $p \le 2$,

$$E|X_n|^p < \infty \qquad (n = 1, 2, ...), \tag{10}$$

$$A_n = \sum_{k=1}^{n} E|X_k|^p \to \infty. \tag{11}$$

In the case $p \ge 1$, suppose that

$$EX_n = 0 \qquad (n = 1, 2, ...). \tag{12}$$

Let $f \in F_c$. If for every positive ε

$$\sum_n P\left(|X_n| > \varepsilon \left(A_{nf}(A_n)\right)^{1/p}\right) < \infty, \tag{13}$$

then

$$S_n = o\left(\left(A_{nf}(A_n)\right)^{1/p}\right) \quad \text{a.s.} \tag{14}$$

Theorem 10. For any function f from F_d and any positive $p \le 2$ there exists a sequence of independent symmetric random variables $\{X_n\}$, satisfying conditions (10), (11), and (13), and for which

$$\limsup |S_n| / \left(A_{nf}(A_n)\right)^{1/p} = \infty \quad \text{a.s.} \tag{15}$$

Theorem 11. For any function f from Ψ_d and any positive $p \le 2$ there exists a sequence of independent symmetric random variables $\{X_n\}$, satisfying conditions (10), (11), and for which relation (15) holds.

4. Let $EX_j^2 < \infty$. Put

$$L_j(y) = \int_{|x|>y} x^2 \, dV_j(x),$$

where $V_j(x)$ is the distribution function of the random variable X_j.

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Theorem 12. If, for some $f \in \Psi_c$, the relation

$$\frac{f(B_n)}{B_n \ln\ln B_n} \sum_{j=1}^{n} L_j \left(\varepsilon \sqrt{\frac{B_n}{\ln\ln B_n}} \right) \to 0 \tag{16}$$

holds for every $\varepsilon > 0$, then the law of the iterated logarithm (l.i.l.) is valid for the sequence $\{X_n\}$.

For any function $f \in \Psi_d$ such that $f(x) < x^{1-\alpha}$ for some $\alpha > 0$ and all sufficiently large x, there exists a sequence of symmetric independent random variables for which

$$EX_n^2 \asymp 1^*, \tag{17}$$

$$\frac{f(B_n)}{B_n \ln\ln B_n} \sum_{j=1}^{n} L_j(1) \to 0, \tag{18}$$

but

$$\limsup \frac{|S_n|}{\sqrt{B_n \ln\ln B_n}} = \infty \quad \text{a.s.} \tag{19}$$

$*$ $a_n \asymp b_n$ means that $0 < \liminf a_n/b_n \leq \limsup a_n/b_n < \infty$.

Theorem 13. If, for some f from F_c, condition (16) is satisfied and

$$\sum_n P\left(|X_n| > \varepsilon\sqrt{B_n \ln\ln B_n}\right) < \infty \tag{20}$$

for every $\varepsilon > 0$, then the LIL holds for $\{X_n\}$.

On the other hand, for any function f from F_d such that $f(x) < x^{1-\alpha}$ for some $\alpha > 0$, there exists a sequence of symmetric independent random variables $\{X_n\}$ for which (20), (18), and (19) are satisfied.

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Leningrad State University
named after A. A. Zhdanov

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REFERENCES

[1] V. V. Petrov, *Probability Theory and Its Applications*, **2**, 193 (1969).

Note: Figure translations are in progress. See original paper for figures.

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