ON LOCALLY CONVEX SPACES WITH A BASIS

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Abstract

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MATHEMATICS

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ON LOCALLY CONVEX SPACES WITH A BASIS

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1. At the present time the theory of basic and minimal systems in Banach spaces is far advanced and finds broad applications in the study of topological properties of spaces (see, for example, the survey ([1])). In broader classes of separable locally convex spaces (l.c.s.) an analogous study of basic systems encounters various obstacles and, as is seen from a number of recent works ([2]-[4]), reveals interesting linear-topological properties of l.c.s. In §§ 2 and 4 of the present note James' theory of bases of Banach spaces (see ([8]), Ch. 4, §§ 3–4) is extended to arbitrary ω-complete (i.e. sequentially complete) l.c.s. with bases. In particular, the well-known theorem of A. Pełczyński ([5]) on B-spaces not containing c_0, as well as the theorem of V. D. Milman ([6]) characterizing the spaces l_p ($1 \le p < \infty$), holds in arbitrary ω-complete l.c.s. Some of the results of §§ 2–3 for a narrower class of l.c.s. are contained in ([2]) and ([3]). In § 5 a criterion is given for the stability of bases in barrelled spaces.

Below, $E(M)$ denotes the closed linear hull of the set M, $B(E)$ denotes the set of all bounded subsets of the space E, and $E_1 \simeq E_2$ means an isomorphism of the spaces E_1 and E_2. Sequences $\{x_k\}_1^\infty \subset E_1$ and $\{y_k\}_1^\infty \subset E_2$ are called equivalent if there exists an isomorphism A of the spaces $E(\{x_k\})$ and $E(\{y_k\})$ such that $Ax_k = y_k$.

2. Let $\{x_k\}_1^\infty$ be a basis of the space E with conjugate system $\{f_k\}_1^\infty \subset E^*$, i.e. a Schauder basis. Define the linear operators $\{U_n\}_1^\infty$ by the formula

$$U_n x = \sum_1^n f_k(x) x_k \quad (n = 1, 2, ...)$$

and denote $E^n = (I - U_n)E$.

We shall need the following analogue of the Banach–Steinhaus theorem (see ([7]), p. 180): a pointwise bounded family of continuous linear mappings from an ω-complete l.c.s. E_1 into an arbitrary l.c.s. E_2 is uniformly bounded on each $M \in B(E_1)$. An immediate consequence of this is

Lemma 1 (see ([4])). If E is ω-complete, then the family $\{U_n\}$ is uniformly bounded on each $M \in B(E)$, i.e. $\{U_n M\} \in B(E)$ and

$$\left\{\sum_1^n f(x_k) f_k\right\}_{n=1}^{\infty} \in B(E^*)$$

for every $f \in E^*$.

We shall call a basis of the space E **shrinking**,* if for any $M \in B(E)$ and $f \in E^*$

$$\sup\{|f(x)| : x \in M \cap E^k\} \to 0$$

as $k \to \infty$.

Theorem 1. Let E be an ω-complete l.c.s. with basis $\{x_k\}_1^\infty$ and conjugate system $\{f_k\}_1^\infty$. The following conditions are equivalent:

1) the basis $\{x_k\}_1^\infty$ is shrinking;

2) the system $\{f_k\}_1^\infty$ is strongly complete in E^*;

3) $\{f_k\}_1^\infty$ is a strong basis in E^*.

Proof. Obviously, 3) implies 2). We show that 1) implies 3). Let $\{x_k\}_1^\infty$ be a shrinking basis, $f \in E^*$ and $M \in B(E)$.

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* Our definition follows the classical definition for Banach spaces, due to James (see ([8]), p. 119). In Theorem 1 the equivalence of this definition with that used in works ([2]) and ([3]) is established.

Put

$$\sum_1^n f(x_k) f_k \quad \text{and} \quad M_n = \{y : \ y = x - U_{kx}, \ x \in M, \ k \geq n\}.$$

By Lemma 1, $M_n \in B(E)$ for all $n = 1, 2, \ldots$. Moreover, $\cdots \supset M_n \supset M_{n+1} \supset \cdots$, and, since $M_n \subset M_1 \cap E^n$,

$$\sup\{|(f - \varphi_n)x| : \ x \in M\} \leq \sup\{|f(x)| : \ x \in M_n\} \to 0, \quad n \to \infty.$$

Thus the series

$$\sum_1^\infty f(x_k) f_k$$

converges strongly to f. It remains to establish that 1) follows from 2). If the system $\{f_k\}_1^\infty$ is strongly complete in E^*, then for arbitrary $M \in B(E)$ and $f \in E^*$ there exists a sequence

$$\varphi_n = \sum a_{nk} f_k$$

such that

$$\sup\{|(f-\varphi_n)x| : x \in M\} \to 0 \quad \text{as } n \to \infty.$$

Since

$$\sup\{|(f-\varphi_n)x| : x \in M\} \geq \sup\{|(f-\varphi_n)x| : x \in M \cap E^n\} = \sup\{|f(x)| : x \in M \cap E^n\},$$

the basis $\{x_k\}$ is shrinking.

A basis $\{x_k\}_1^\infty$ of a space E is called **boundedly complete** if, from the boundedness of the partial sums

$$\left\{ y_n = \sum_1^n a_{kx} k \right\}$$

it follows that $y_n \to x \in E$ as $n \to \infty$.

Let us note that if E is ω-complete and $\{f_k\}_1^\infty$ is a boundedly complete basis in $E(\{f_k\}_1^\infty)$, then $\{x_k\}_1^\infty$ is a shrinking basis in E*. Indeed, it is easy to see that $\{f_k\}_1^\infty$ is a strong basis in all of E^* and, consequently, $\{x_k\}_1^\infty$ is a shrinking basis.

Theorem 2. *A basis of an ω-complete l.c.s. E is simultaneously shrinking and boundedly complete if and only if E is semireflexive**.*

Proof. Let E be semireflexive. By the Hahn–Banach theorem, $\{f_k\}$ is a complete system in E^*, i.e. $\{x_k\}$ is a shrinking basis. Let the sequence of numbers $\{a_k\}_1^\infty$ be such that

$$\left\{ y_n = \sum_1^n a_{kx} k \right\}_{n=1}^\infty \in B(E).$$

The set

$$S = \{y_p - y_q\}_{p>q=1}^\infty \in B(E).$$

For any $f \in E^*$ and $n \geq m$,

$$|f(y_n - y_m)| \leq \sup\{|f(x)| : \ x \in S \cap E^m\} \to 0, \quad m \to \infty,$$

since $y_n - y_m \in S \cap E^m$, and the basis $\{x_k\}_1^\infty$ is shrinking. A semireflexive space is weakly ω-complete ((7), p. 239); consequently,

$$y_n \to x_0 \in E$$

(weakly). Then $f_k(x_0) = a_k$, and

$$\sum_1^\infty a_{kx} k = \sum_1^\infty f_k(x_0) x_k = x_0.$$

The converse part of the proof is carried out in the same way as in ([2]).

3. We shall need the following

Lemma 2. *Let E_1 be a barrelled space with basis $\{x_k\}_1^\infty$ and conjugate system $\{f_k\}_1^\infty$, and let the space E_2 be ω-complete. If $\{y_k\}_1^\infty \subset E_2$ and*

$$\left\{\sum_1^n f_k(x) y_k\right\}_{n=1}^\infty \in B(E_2)$$

for all $x \in E_1$, then the series $\sum_1^\infty f_k(x) y_k$ converges for all $x \in E_1$, and the linear operator $A : E_1 \to E_2$, where

$$Ax = \sum_1^\infty f_k(x) y_k,$$

is continuous.

For the proof of the lemma it is enough to note that the family of operators $\{S_n : E_1 \to E_2\}_1^\infty$, where

$$S_{nx} = \sum_1^n f_k(x) y_k,$$

is equicontinuously continuous.

* The dual assertion for barrelled l.c.s. is found in ([3]). Without additional boundedness assumptions the dual assertion is not true.

** The case of barrelled spaces is treated in ([2]).

Denote by E_η the space E in the strongest locally convex topology η such that $B(E_\eta) = B(E)$. With each absolutely convex set $M \in B(E)$ we shall associate the normed space E_M, $E_M = \{x \in E : \lambda^{-1}x \in M$ for some $\lambda > 0\}$, and $\|x\|_M = \inf\{\lambda : \lambda^{-1}x \in M\}$. If E is ω-complete and M is closed, then E_M is a Banach space (see ([7]), pp. 158 and 181). The space E_η is the inductive limit of the spaces E_M; here the family $\{M\}$ is regarded as ordered by inclusion (see ([7]), p. 181).

Theorem 3. *Let a sequence $\{x_k\}_1^\infty$ in an ω-complete l.c.s. E be such that, for some $p \geq 1$, the series*

$$\sum a_{kx}k$$

converges if and only if

$$\sum_1^\infty |a_k|^p < \infty.$$

Then E contains a subspace E_1 such that $(E_1)_\eta \simeq l_p$, and $\{x_k\}_1^\infty$ is equivalent to the natural basis of l_p after the removal of a certain finite number of elements.

Remark. For Banach spaces this theorem is due to V. D. Milman ([6]).

Proof of the theorem. Denote by D the unit ball of l_p, and by $\{e_k\}_1^\infty$ the natural basis in l_p. By Lemma 2 the linear operator $A : l_p \to E$, $Ae_k = x_k$, is continuous, $AD \in B(E)$, and $\{x_k\}_1^\infty \subset AD$. Let $M \in B(E)$, and let the space E_M associated with M be Banach. If $AD \subset M$, then A is continuous from l_p to E_M. Denote by E_1 the closed linear span of $\{x_k\}$ in E_M. With the aid of Lemma 2 it is not difficult to verify that the hypotheses of V. D. Milman' s theorem ([6]) are fulfilled, whence $E_1 \simeq l_p$ and $\{x_k\}$ is a natural basis of E_1 after the removal of a certain finite number of elements. Thus, $M \cap E_1$ absorbs the set AD, which proves the theorem.

We note that for $p > 1$ the subspace E_1 is sequentially closed in E.

4. In the class of Banach spaces, spaces not containing a c_0-subspace are distinguished by many interesting properties. This is connected primarily with A. Pełczyński' s theorem ([5]), which, as shown in Theorem 4, holds in an arbitrary ω-complete l.c.s.

Theorem 4. *If a sequence $\{x_k\}_1^\infty$ in an ω-complete l.c.s. E is such that, for all $f \in E^*$, the series*

$$\sum^\infty |f(x_k)|$$

converges, while the series

$$\sum x_k$$

does not converge strongly, then E contains a closed subspace $E_1 \simeq c_0$.[1]

Proof. There exist a seminorm $p(x)$, continuous on E, and a sequence of numbers $\{n_k\}_{k=1}^{\infty}$ such that

$$p\left(\sum_{n_k+1}^{n_{k+1}} x_k\right) \geq \varepsilon > 0.$$

Putting

$$\sum_{n_k+1}^{n_{k+1}} x_k = y_k,$$

for any $f \in E^*$ we obtain $f(y_k) \to 0$ $(k \to \infty)$, while $p(y_k) \geq \varepsilon > 0$. With the aid of the theorem of Bessaga and Pełczyński ([9]), it is not difficult to show that from the sequence $\{y_n\}$ one can select a uniformly minimal sequence $\{y_{n_k}\}_{k=1}^{\infty}$ with conjugate $\{f_k\}_1^{\infty}$ $(f_k(y_{n_j}) = \delta_{kj})$, i.e. one such that $|f_k(x)| \leq \delta p(x)$ for some $\delta > 0$ for all $x \in E(\{y_{n_k}\}_{k=1}^{\infty})$. If $f \in E^*$ and the sequence of numbers $a_k \to 0$, then

$$\sum^{\infty} |f(a_k y_{n_k})| < \infty;$$

hence,

$$\left\{\sum_{k=1}^{m} a_k y_{n_k}\right\}_{m=1}^{\infty} \in B(E).$$

Since E is ω-complete, by Lemma 2 there exists $C > 0$ such that,

for any $(a_k) \in c_0$:

$$C \max |a_k| \geq p'\left(\sum_{k=1}^{\infty} a_k y_{\eta_k}\right) \geq \frac{1}{\delta} \max |a'_k|.$$

Thus the linear operator $A : c_0 \to E$, defined on the natural basis $\{e_k\}$ of the space c_0 by the formula $Ae_k = y_{\eta_k}$ $(k = 1, 2, ...)$, is continuous, extends to all of c_0, and has a continuous inverse on its range E_1. Since c_0 is a complete space, E_1 is closed in E and

$$E_1 = E(\{y_{n_k}\}_{k=1}^{\infty}) \simeq c_0.$$

Corollary. If an unconditional basis of an ω-complete space E is not boundedly complete, then E contains c_0.

For barrelled spaces this corollary and Theorems 5 and 6 were established in ([3]). However, for barrelled spaces it is not necessary to pass to the η-topology, as is done in the theorems below. For arbitrary locally convex spaces such a passage is already necessary.

[1] As we have learned, this fact was obtained simultaneously by D. B. Dimitrov (unpublished).

Theorem 5. If an unconditional basis of an ω-complete space E is not total, then E contains a subspace E_1 such that $E_{1\eta} \simeq l_1$.

The proof is close to the proof of Theorem 4, but also uses Theorem 3 with $p = 1$ and Lemma 1.

Combining the results of Theorems 4 and 5, we obtain:

Theorem 6. If an ω-complete l.c.s. E with an unconditional basis is not semireflexive, then it contains a subspace E_1 such that either $E_1 \simeq c_0$, or $E_{1\eta} \simeq l_1$.

Remark. In all the results given above (except Lemma 2) one may weaken the requirement of ω-completeness of the space E, requiring only its **local completeness**.* The assertion of Lemma 2 remains valid if E_1 is Banach and E_2 is locally complete.

5. In conclusion we state, without proof, one theorem on the stability of a Schauder basis in a barrelled space, analogous to V. D. Milman' s criterion [10] for bases of Banach spaces.

Theorem 7. Let $\{x_k\}_1^\infty$ be a Schauder basis in an ω-complete barrelled space E with conjugate system $\{f_k\}_1^\infty$. If the series

$$\sum^\infty \varepsilon_k f_k$$

converges weakly* unconditionally in E^*, then for every $M \in B(E)$ there exists $\delta_0 > 0$ such that, for all $\delta < \delta_0$ and $\{y_k\}_1^\infty \subset M$, the sequence $\{x_k + \delta\varepsilon_k y_k\}_1^\infty$ is a basis of E, equivalent to $\{x_k\}_1^\infty$. Conversely, if for every $M \in B(E)$ there exists $\delta_0 > 0$ with the properties indicated above, then the series

$$\sum^\infty \varepsilon_k f_k$$

converges weakly* unconditionally in E^*.

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* The space E is called **locally complete** if every $M \in B(E)$ is contained in some $N \in B(E)$ such that the space E_N associated with N is Banach.

Note: Figure translations are in progress. See original paper for figures.