THE METHOD OF FUNCTIONALS AS APPLIED TO PROBLEMS OF THE ZOLOTAREV-PSHEBORSKY TYPE

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Abstract

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THE METHOD OF FUNCTIONALS AS APPLIED TO PROBLEMS OF THE ZOLOTAREV-PSHEBORSKY TYPE

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The Zolotarev ([1])–Psheborsky ([2]) problem in its general form is formulated as follows. Among $\{P_n(x)\}$ satisfying two relations

$$\sum_0^n p_i\mu_i = A$$

($\neq 0$) and

$$\sum_0^n p_i\nu_i = B$$

find $Y_n(x)$, the one least deviating from zero on $[0,1]$, and its deviation L.

In 1955 we proved the following theorem ([3]).

Let $F_1 = \mu_0, \mu_1, \dots, \mu_n$ and $F_2 = \nu_0, \nu_1, \dots, \nu_n$ be given linear functionals on polynomials of degree n, and

$$F_3 = \mu_0 + \Omega\nu_0, \quad \mu_1 + \Omega\nu_1, \dots, \mu_n + \Omega\nu_n \tag{1}$$

for $-\infty < \Omega < +\infty$. If F_3 is served by the family of polynomials $\{Q_n(x,\Omega)\}$, then the necessary and sufficient condition that $Y_n(x)/L \in \{Q_n(x,\Omega)\}$ consists in the following: the equation

$$F_2[Q_n(x,\Omega)] = \lambda F_1[Q_n(x,\Omega)], \tag{2}$$

where $\lambda = B/A$, must have a real root $\Omega = \Omega_0$; then

$$Y_n(x) = LQ_n(x,\Omega_0)$$

and

$$1/L = F_1[Q_n(x, \Omega_0)]/A = F_2[Q_n(x, \Omega_0)]/B. \tag{3}$$

The number of solutions of the problem is equal to the number of the roots mentioned.

Here we shall consider the application of this theorem in certain special cases, but first we shall prove some general assertions concerning the functional (1).

Let there be given: the segment-functional $\nu_0, \nu_1, \dots, \nu_n$ and a system of arbitrary nodes $(\sigma_i)_1^s$, where $0 \le \sigma_1 < \sigma_2 < \cdots \sigma_s \le 1$; $s \le n+1$. The condition for the decomposability of the segment with respect to these nodes is the condition of compatibility of the system of $n+1$ equations with s unknowns (Δ):

$$\sum_{i=1}^{s} \Delta_i \sigma_i^k = \nu_k, \tag{4}$$

$(k = 0, 1, \dots, n)$; in other words, the unique system of solutions (Δ_i) obtained from the first s equations must also satisfy the remaining ones (in the case $s = n+1$ a decomposition is always possible).

Introduce the notation:

$$\prod_1 (x - \sigma_i) = R_s(x)$$

—the resolvent of the system of nodes—and

$$r_0 \nu_k + r_1 \nu_{k+1} + \cdots + r_{s-1} \nu_{k+s-1} + \nu_{k+s} \equiv R_s(\nu_k^{k+s})$$

for $k + s \le n$. Then the compatibility condition can be formulated in the following form:

Theorem 1. The necessary and sufficient condition for the decomposability of $(\nu_k)_0^n$ with respect to any preassigned nodes $(\sigma_i)_1^s$, belonging to $[0, 1]$

for $s < n+1$, are as follows: the equalities

$$R_s(\bar{v}_0^s) = 0, \qquad R_s(\bar{v}_1^{s+1}) = 0, \dots, \qquad R_s(\bar{v}_{n-s}^n) = 0. \tag{5}$$

must hold.

Let the system (4) be consistent, i.e.

$$v_k = \sum_1^s \Delta_i \sigma_i^k \qquad (k = 0, 1, \dots, n).$$

Then

$$R_s(\bar{v}_0^s) = \sum_0^s r_k v_k = \sum_0^s r_k \sum_1^s \Delta_i \sigma_i^k = \sum_1^s \Delta_i \sum_0^s r_k \sigma_i^k = \sum_1^s \Delta_i R_s(\sigma_i) = 0,$$

and, in general,

$$R_s(\bar{v}_l^{l+s}) = \sum_0^s r_k v_{l+k} = \sum_0^s r_k \sum_1^s \Delta_i \sigma_i^{l+k} = \sum_1^s \Delta_i \sigma_i^l \sum_0^s r_k \sigma_i^k = \sum_1^s \Delta_i \sigma_i^l R_s(\sigma_i) = 0.$$

Suppose the conditions (5) are fulfilled; the numbers $v_0, \dots, v_{s-1}$ always have the structure (4). Assume that v_s does not have this structure, i.e.

$$v_s = \sum_1^s \Delta_j \sigma_j^s + A \quad (A \neq 0).$$

Then

$$R_s(\bar{v}_0^s) = \sum_{k=0}^s r_k \sum_{j=1}^s \Delta_j \sigma_j^k + A = \sum_1^s \Delta_j R_s(\sigma_j) + A \neq 0,$$

which contradicts the condition; consequently, v_s has the same structure; in the same way we verify the consistency of the entire system (4).

Thus, the possibility of expanding $(v_k)_0^n$ with respect to the nodes $(\sigma_i)_1^s$ requires that the segment $(v_k)_0^{s-1}$ be extended by the "resolvent" of the system $(\sigma_i)_1^s$.

Remark 1. The expandability of $(v_k)_0^n$ with respect to $(\sigma_i)_1^s$ creates, after the determination of (Δ_i), a distribution $(\sigma_i)_1^s$, which is true for $(v_k)_0^n$ if and only if there exists a reduced polynomial $Q_n(x)$ with this distribution (extremal or serving $(v_k)_0^n$). The theorem remains valid for arbitrary $-\infty < (\sigma_i)_1^s < +\infty$.

Theorem 2. If $(\mu_k)_0^n$ has $Q_n(x)$ as its extremal polynomial, then the necessary and sufficient conditions for $Q_n(x)$ to be extremal also for $(\mu_k + v_k)_0^n$ are: 1) the nodes of the distribution of $Q_n(x)$, $(\sigma_i)_1^s$, are suitable for $(v_k)_0^n$, i.e. $R_s(\bar{v}_k^{k+s}) = 0$, and 2) if $(\Delta_i)_1^s$ are the solutions of the system (4) for (v_k), and $(\delta_i)_1^s$ are the solutions of (4) for $(\mu_k)_0^n$, then

$$\operatorname{sgn}(\delta_i + \Delta_i) = \operatorname{sgn} \delta_i \qquad (i = 1, 2, \dots, s). \tag{6}$$

Let $Q_n(x)$ be extremal also for $(\mu_k + v_k)_0^n$; then the system (4) is consistent also for $(\mu_k + v_k)_0^n$, i.e. (Theorem 1)

$$R_s(\overline{\mu + v}_k^{k+s}) = R_s(\bar{\mu}_k^{k+s}) + R_s(\bar{v}_k^{k+s}) = 0,$$

whence also $R_s(\bar{v}_k^{k+s}) = 0$. Further, the loads (the roots of the system (4)) for $(\mu_k + v_k)_0^n$ are $\Delta_i + \delta_i$, and then for the extremality of $Q_n(x)$ it is necessary that

$$\operatorname{sgn}(\Delta_i + \delta_i) = \operatorname{sgn}\delta_i.$$

Let $R_n(\bar{v}_k^{k+s}) = 0$; then the system (4) is consistent for $(v_k)_0^n$, consequently it is consistent also for $(\mu_k + v_k)_0^n$, and if the conditions (6) are fulfilled, then, according to the extremality criterion (3), $Q_n(x)$ serves $(\mu_k + v_k)_0^n$.

Theorem 3. The necessary and sufficient condition for an interval operator of the form (1) to have, on Ω, a critical interval (3), and moreover a unique one (Ω', Ω''), is the following: $(v_k)_0^n$ is served by one of the Chebyshev polynomials $T_n(x) = \cos n \arccos(2x - 1)$ or $-T_n(x)$.

Let the true distribution of $(v_k)_0^n$ be $(\sigma_i)_1^s$, not coinciding with the Chebyshev distribution $(\tau_i)_0^n$; the same or the opposite $(\sigma_i)_1^s$ is possessed also by $(\Omega v_k)_0^n$ for any $\Omega \lessgtr 0$, and, consequently, the expansion of $(\mu_k + \Omega v_k)_0^n$ with respect to $(\tau_i)_0^n$ is always fictitious. Meanwhile, for sufficiently large $|\Omega|$, the always possible expansion of $(\mu_k + \Omega v_k)_0^n$ with respect to $(\tau_i)_0^n$ will have signs of the loads coincid-

giving, with the signs of the loads for $(\Omega\nu_k)_0^n$ at the same $(\tau_i)_0^n$, i.e. $\pm T_n(x)$, are unsuitable for serving (1).

Let $(\nu_k)_0^n$ be served by $+T_n(x)$; then $(\Omega\nu_k)_0^n$ is served by one of the $\pm T_n(x)$. Whatever the decomposition of $(\mu_k)_0^n$ over the nodes $(\tau_i)_0^n$, for sufficiently large $|\Omega|$ we have, for $(\mu_k + \Omega\nu_k)_0^n$, when $\Omega > 0$, the extremal $+T_n(x)$, and when $\Omega < 0$, it is $T_n(x)$. The theorem is proved.

Remark 2. This theorem admits the following extension. Let $Q_n(x)$, with distribution $(\sigma_i)_1^s$, serve the interval $(\nu_k)_0^n$; if the interval $(\mu_k)_0^n$ is decomposed over the nodes $(\sigma_i)_1^s$ (see the conditions of Theorem 1), then for sufficiently large $|\Omega|$ the polynomials $\pm Q_n(x)$ remain extremal also for $(\mu_k + \Omega\nu_k)_0^n$.

We pass to the study of concrete Zolotarev–Pszeborski problems.

Problem 1. We seek the aforementioned $Y_n(x)$ and its deviation L from $\{P_n(x)\}$, satisfying the conditions $P_n(\rho_1) = A$ and $P_n(\rho_2) = B$, where $1 < \rho_1 < \rho_2$; $A \neq 0$. Here $(\mu_k)_0^n = 1, \rho_1, \rho_1^2, \dots, \rho_1^n$; $(\nu_k)_0^n = 1, \rho_2, \dots, \rho_2^n$. We form the functional (1) (replacing Ω by $-\Omega$)

$$1 - \Omega,\ \rho_1 - \Omega\rho_2, \dots, \rho_1^n - \Omega\rho_2^n, \quad -\infty < -\Omega < +\infty. \tag{7}$$

According to Theorem 3, the interval (7) has a unique critical interval. Its endpoints Ω' and Ω'' are found from the formulas for the loads δ_k (when decomposing the interval (7) over the nodes $(\tau_i)_0^n$ [(3)]). We have, denoting

$$R_{n+1}(x) = \prod_0^n (x - r_i)$$

—the resolvent of $(\tau_i)_0^n$, and $R_n^{(k)}(x) = R_{n+1}(x)/(x - \tau_k)$,

$$\Omega' = R_n^{(0)}(\rho_1)/R_n^{(0)}(\rho_2); \quad \Omega'' = R_n^{(n)}(\rho_1)/R_n^{(n)}(\rho_2)$$

(obviously, $0 < \Omega' < \Omega'' < 1$). Thus, for $\Omega = \Omega'$, δ^0 drops out, and for $\Omega = \Omega''$, δ_n drops out. In view of the fact that (7) has a fictitious two-node structure, the number of its true nodes $s \geq n$ for every Ω. Hence, in the critical interval (7) possesses Zolotarev stability [(4)], and by the theorem on continuous deformation [(5)] it is served by all and only the polynomials $Z_n(x, \vartheta)$ of passport $[n, n, 0]$, which, as Ω varies from Ω' to Ω'', deform from $-T_n(x)$, with the load of the node $\tau_0 = 0$ absent, to $+T_n(x)$, with the load of the node $\tau_n = 1$ absent.

In the family $Z_n(x, \vartheta)$, ϑ is the leading coefficient; it is connected with Ω one-to-one, namely we have:

$$Z_n(\rho_1, \vartheta) - \Omega Z_n(\rho_2, \vartheta) = \max(\vartheta). \tag{8}$$

Since for these polynomials [(3)] $\partial Z_n/\partial\vartheta = \widetilde{R}_n(x, \vartheta)$, where

$$\widetilde{R}_n(x, \vartheta) = \prod_1^n [x - \sigma_i(\vartheta)]$$

is the resolvent of the distribution $Z_n(x, \vartheta)$, it follows from (8) that

$$\widetilde{R}_n(\rho_1, \vartheta) - \Omega\widetilde{R}_n(\rho_2, \vartheta) = 0, \quad \Omega = \widetilde{R}_n(\rho, \vartheta)/\widetilde{R}_n(\rho_2, \vartheta). \tag{9}$$

Thus each polynomial from $\{Z_n(x, \vartheta)\}$ serves (7) at exactly one point Ω; conversely, at each point Ω the interval (7) is served by a unique polynomial, since (7) belongs to class II. Consider now equation (2). We have (upon passing from Ω to ϑ)

$$Z_n(\rho_2, \vartheta) = \lambda Z_n(\rho_1, \vartheta), \tag{10}$$

where $\lambda = B/A$, $-2^{2n-1} \leq \vartheta \leq 2^{2n-1}$, and $-T_n(x) \leq Z_n(\rho, \vartheta) \leq T_n(x)$ $(\rho > 1)$. In view of these bounds, the two curves in (10) necessarily intersect for any λ; this intersection is unique, except at the very endpoints.

Indeed, since $\partial Z_n(\rho,\vartheta)/\partial\vartheta = R_n(\rho,\vartheta) > 0$, $Z_n(\rho,\vartheta)$ increases monotonically in ϑ; moreover, since

$$R_n(\rho,\vartheta) = \prod_1^n [\rho - \sigma_i(\vartheta)]$$

and we have proved [3] that, as ϑ increases, the nodes shift to the left, then

$R_n(\rho,\vartheta)$ also increases monotonically with ϑ. Thus, $Z_n(\rho,\vartheta)$ is concave, and the intersection of the curves (10) inside the critical interval is unique.

Finally, consider

$$\lambda(\vartheta) = Z_n(\rho_2,\vartheta)/Z_n(\rho_1,\vartheta) \tag{11}$$

for $-\infty < \lambda_2 < +\infty$. A violation of uniqueness occurs for $\vartheta = \pm 2^{2n-1}$, i.e., for $\lambda = T_n(\rho_2)/T_n(\rho_1) = -T_n(\rho_2)/-T_n(\rho_1)(> 1)$. Since $\{Z_n(x,\vartheta)\}$ does not contain polynomials of the form $-Z_n(x,\vartheta)$, there are no other violations of uniqueness. $\lambda(\vartheta)$ has a discontinuity at the root ϑ_0 of the equation $Z_n(\rho_1,\vartheta) = 0$: as $A \to 0\pm$, $\lambda \to \pm\infty$; otherwise $\lambda(\vartheta)$ decreases monotonically from $T_n(\rho_2)/T_n(\rho_1)$ to $-\infty$ and from $+\infty$ to $T_n(\rho_2)/T_n(\rho_1)$. Note that $\vartheta_0 \neq 0$, since $Z_n(x,0) = -T_{n-1}(x)$ and for $x > 1$ does not intersect the ϑ-axis.

Thus, the posed problem for $\lambda = T_n(\rho_2)/T_n(\rho_1)$ gives the unique $Y_n(x) = LZ_n(x,\vartheta^*)$, where ϑ^* is the root of equation (10),

$$1/L = Z_n(\rho_2,\vartheta^*)/B = Z_n(\rho_1,\vartheta^*)/A.$$

Problem 2. The initial conditions for determining $Y_n(x)$ and L are as follows:

$$\sum_a^n p_i\mu_i = A \quad \text{and} \quad p_n = B.$$

It is more convenient to replace these conditions by the equivalent ones

$$\sum_0^{n-1} p_i\mu_i = A_1(= A - B\mu_n) \quad \text{and} \quad p_n = B.$$

Then

$$F_1 = \mu_0,\mu_1,\dots,\mu_{n-1},0, \qquad F_2 = 0_0,0_1,\dots,0_{n-1},1,$$

$$F_3 = \mu_0,\mu_1,\dots,\mu_{n-1},\Omega. \tag{12}$$

According to theorem 3, there exists a definite critical interval (Ω', Ω''), outside which F_3 is served by the polynomials $\pm T_n(x)$. For $\Omega' \le \Omega \le \Omega''$, the segment (12) is of class II, except perhaps at a single point (the segment $\mu_0, \dots, \mu_{n-1}$ is assumed not to be absolutely monotone). Let $Q_n(x, \Omega) = q_n(\Omega)x^n + \dots + q_0(\Omega)$ be the extremal polynomial of F_3; then its norm is $N(\Omega) = q_n(\Omega)\Omega + F_2[Q_n(x, \Omega)]$. Equation (2) here is:

$$\lambda F_2[Q_n(x, \Omega)] = q_n(\Omega) \quad \text{or} \quad N(\Omega) = q_n(\Omega)(\Omega + 1/\lambda). \tag{13}$$

As was proved in ([3]), $N(\Omega)$ has one minimum at the focus $\Omega = \Omega^*$, decreasing monotonically to the left of Ω^* and increasing monotonically to the right. For $\Omega < \Omega'$ and for $\Omega > \Omega''$, N is a linear function. Further, $q_n(\Omega) = dN/d\Omega$ and increases monotonically in (Ω', Ω'') from -2^{2n-1} to $+2^{2n-1}$, remaining constant outside the critical interval. The existence of an intersection of the curves in (13), i.e., the possibility of solving the problem by the proposed method, always occurs when $N(\Omega') < 2^{2n-1}(\Omega'' + 1/\lambda)$, and then the solution is two-valued; one of them is

$$Y_n(x) = \frac{B}{2^{2n-1}} T_n(x).$$

For special $(\mu_k)_0^n$, cases are possible in which this method is inapplicable.

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Note: Figure translations are in progress. See original paper for figures.