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Abstract

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MATHEMATICS

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A NORMALLY SOLVABLE EQUATION OF SMOOTH TRANSITION

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Many problems of mathematical physics lead to a "paired" integral equation, which in the normal case can be written in the form

$$f(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_1(x-t)f(t)\,dt-g(x)=0,\quad x>0,$$

$$f(t)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_2(x-t)f(t)\,dt-g(x)=0,\quad x<0. \tag{1}$$

The theory of equation (1) was constructed in the works of I. M. Rapoport ([1]), I. Ts. Gokhberg and M. G. Krein ([2]), and other authors. A "paired" equation describes a process determined by different conditions for $x>0$ and $x<0$, with an explicitly expressed point of separation $x=0$.

It is of interest to consider the case when the above-mentioned conditions *gradually* pass one into the other. The corresponding integral equation may be taken in the form

$$f(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_1(x-t)f(t)\,dt-g(x)+$$

$$+e^{-x}\left\{f(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_2(x-t)f(t)\,dt-g(x)\right\}=0,\quad -\infty<x<\infty. \tag{2}$$

We shall call this integral equation the equation of smooth transition. Equation (2) can be put into the form

$$f(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}n_1(x-t)f(t)\,dt+\frac{\operatorname{th}(x/2)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}n_2(x-t)f(t)\,dt=g(x),$$

$$-\infty < x < \infty, \tag{3}$$

where $2n_1(x) = k_1(x) + k_2(x)$, $2n_2(x) = k_1(x) - k_2(x)$.

Assume that the kernel functions $k_1(x)$ and $k_2(x)$ are summable on the whole axis. Then, according to a result of L. S. Rakovshchik (see, for example, ([3]), p. 274), in order that equation (3) be normally solvable in the space $L_2(-\infty, \infty)$ and have a finite index, it is necessary and sufficient that

$$1 + K_1(x) \neq 0, \quad 1 + K_2(x) \neq 0, \tag{4}$$

where

$$K_j(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_j(t) e^{ixt}\, dt$$

is the Fourier integral of the kernel function $k_j(x)$.

It is shown below that, similarly to equation (1), the equation of smooth transition can be solved by quadratures.

Suppose that conditions (4) are satisfied and $g(x) \in L_2(-\infty, \infty)$. Assuming that a solution of equation (2) exists in the space $L_2(-\infty, \infty)$, introduce a new unknown function

$$\varphi(x) = f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_2(x-t) f(t)\, dt - g(x), \tag{5}$$

which, by virtue of (2), has the properties

$$\varphi(x) \in L_2(-\infty, \infty), \qquad e^{-x}\varphi(x) \in L_2(-\infty, \infty). \tag{6}$$

Theorem 1 ((4), Chapter V). In order that the function $\varphi(x)$ satisfy conditions (6), it is necessary and sufficient that its Fourier integral $\Phi(x)$ be analytically continuable to the strip $0 < \operatorname{Im} z < 1$, and moreover uniformly with respect to y

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^2\, dx < \text{const}, \qquad 0 \leq y \leq 1.$$

This theorem makes it possible to reduce equation (2) to a Carleman boundary-value problem for a strip; for this it is enough to pass in equalities (2) and (5) to Fourier integrals and eliminate the unknown function $F(x)$:

$$\Phi(x) = -\frac{1+K_2(x)}{1+K_1(x)}\Phi(x+i) + \frac{K_2(x)-K_1(x)}{1+K_1(x)}G(x), \qquad -\infty < x < \infty. \quad (7)$$

Here the unknown is the function Φ, which must satisfy the corresponding conditions of Theorem 1.

To reduce the Carleman problem (7) to a Riemann problem, introduce the function

$$\omega(z) = \frac{1}{\sqrt{z}}\Phi\left(\frac{\ln z}{2\pi}\right). \quad (8)$$

Here the functions $\ln z$ and $1/\sqrt{z}$ are defined and analytic in the complex plane cut along the positive part of the real axis. On the upper edge of the cut the logarithm takes real values, and the square root positive values.

Theorem 2. In order that the function $\Phi(x)$ satisfy the conditions of Theorem 1, it is necessary and sufficient that function (8) be representable by a Cauchy-type integral

$$\omega(z) = \frac{1}{2\pi i}\int_0^\infty \frac{\Omega(\tau)\,d\tau}{\tau - z}, \quad (9)$$

where $\Omega(\tau) \in L_2(0,\infty)$.

Putting $\xi = e^{2\pi x}$ and using equality (8), we give the boundary condition (7) the form

$$\omega^+(\xi) = \frac{1+K_2(\ln\xi/2\pi)}{1+K_1(\ln\xi/2\pi)}\omega^-(\xi) + H(\xi), \qquad \xi > 0. \quad (10)$$

Here $\omega^+(\xi)$ and $\omega^-(\xi)$ are the limiting values of the sought analytic function (9), respectively on the upper and lower edge of the cut, and $H(\xi)$ is a known function belonging to $L_2(0,\infty)$,

$$H(\xi) = \frac{K_2(x)-K_1(x)}{1+K_1(x)}G(x)e^{-\pi x}. \quad (11)$$

As will be shown below, problem (10) is a special case of the following Riemann problem on a line: find the limiting values $\omega^+(\xi)$ and $\omega^-(\xi)$ of a function representable by a Cauchy-type integral

$$\omega(z) = \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{\Omega(\tau)\,d\tau}{\tau - z}, \qquad \Omega(\tau) \in L_2(-\infty,\infty),$$

satisfying the condition

$$\omega^+(\xi) = A(\xi)\omega^-(\xi) + H(\xi), \qquad -\infty < \xi < \infty. \tag{12}$$

Here $A(\xi)$ is a given continuous function on the closed axis, satisfying the condition $A(\xi) \neq 0$; $H(\xi)$ is a given function belonging to $L_2(-\infty, \infty)$.

Theorem 3. Let $\varkappa$ be the index of the function $A(\xi)$

$$\varkappa = \frac{1}{2\pi} \arg A(\xi)\Big|_{-\infty}^{+\infty}.$$

Then, for $\varkappa \geq 0$, problem (12) is solvable for any function $H(\xi)$ from $L_2(-\infty, \infty)$. The general solution contains $\varkappa$ linearly independent components and has the form

$$\omega^\pm(\xi) = X^\pm(\xi)\left[\pm\frac{H(\xi)}{2X^+(\xi)} + \frac{\xi + i}{2\pi i}\int_{-\infty}^{\infty} \frac{H(\tau)\,d\tau}{X^+(\tau)(\tau+i)(\tau-\xi)} + \frac{P_{\varkappa-1}(\xi)}{(\xi+i)^\varkappa}\right], \tag{13}$$

where

$$X^+(\xi) = \exp\left\{\frac{1}{2}\ln\left[A(\xi)\left(\frac{\xi+i}{\xi-i}\right)^\varkappa\right] + \frac{\xi+i}{2\pi i}\int_{-\infty}^{\infty} \ln\left[A(\tau)\left(\frac{\tau+i}{\tau-i}\right)^\varkappa\right]\frac{d\tau}{(\tau+i)(\tau-\xi)}\right\},$$

$$X^-(\xi) = \left(\frac{\xi+i}{\xi-i}\right)^\varkappa \exp\left\{-\frac{1}{2}\ln\left[A(\xi)\left(\frac{\xi+i}{\xi-i}\right)^\varkappa\right] + \frac{\xi+i}{2\pi i}\int_{-\infty}^{\infty} \ln\left[A(\tau)\left(\frac{\tau+i}{\tau-i}\right)^\varkappa\right]\frac{d\tau}{(\tau+i)(\tau-\xi)}\right\}.$$

$P_{\varkappa-1}(\xi)$ is an arbitrary polynomial of degree not exceeding $\varkappa - 1$; $\ln\left[A(\xi)\left(\frac{\xi+i}{\xi-i}\right)^\varkappa\right]$ is a continuous function. In the case $\varkappa < 0$, the following conditions are necessary and sufficient for the solvability of problem (12):

$$\int_{-\infty}^{\infty} \frac{H(\xi)(\xi-i)^{k-1}}{X^+(\xi)(\xi+i)^{k+1}}\,d\xi = 0, \qquad k = 1, 2, \dots, -\varkappa. \tag{14}$$

For $\varkappa \leq 0$, the solution of the problem is unique and is given by formulas (13), where $P_{\varkappa-1}(\xi) \equiv 0$. Singular integrals are understood in the sense of the principal value.

The proof of the theorem consists in passing, by means of a linear-fractional transformation, to the Riemann problem on a circle and comparing the results of V. V. Ivanov ([5]), I. B. Simonenko ([6]), and B. V. Khvedelidze ([7]).

To bring problem (12) to the form (10), it is enough to set $H(\xi) = 0$ for $\xi < 0$ and

$$A(\xi) = \begin{cases} \left[1 + K_2\left(\frac{\ln \xi}{2\pi}\right)\right]\left[1 + K_1\left(\frac{\ln \xi}{2\pi}\right)\right]^{-1}, & \xi > 0, \\ 1, & \xi < 0. \end{cases} \tag{15}$$

The function (15) has no zeros and is continuous on the closed axis by assumption (4) and by the properties of the Fourier integrals of summable functions. The index of the function (15) is equal to

$$\varkappa = \frac{1}{2\pi} \arg[1 + K_2(\xi)]\Big|_{-\infty}^{+\infty} - \frac{1}{2\pi} \arg[1 + K_1(\xi)]\Big|_{-\infty}^{+\infty}.$$

Noting that, in passing from equation (2) to the Riemann problem (10), only invertible operations were used, we complete the proof of the main theorem.

Theorem 4. Let $k_1(x) \in L$, $k_2(x) \in L$, and suppose that conditions (4) are satisfied. If the index χ of the function (15) is positive, then the homogeneous equation (2) has in the space L_2 exactly χ linearly independent solutions. For the solvability of the nonhomogeneous equation (2) in this space it is necessary and sufficient that $g(x) \in L_2$.

In the case $\chi = 0$, the homogeneous equation has in L_2 only the zero solution; for the solvability of the nonhomogeneous equation in L_2 it is necessary and sufficient that $g(x) \in L_2$.

For $\chi < 0$, the homogeneous equation has in L_2 only the zero solution. For the solvability of the nonhomogeneous equation (2) in the space L_2 it is necessary and sufficient that the free term $g(x)$ belong to L_2 and annihilate $|\chi|$ functionals in accordance with conditions (14).

In all cases the solution of the equation of smooth transition is constructed in quadratures:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[G(t) + e^{\pi t}\omega^+(e^{2\pi t})\right] \frac{e^{-ixt}\,dt}{1 + K_2(t)},$$

where $G(t)$ and $K_2(t)$ are the Fourier integrals of the functions $g(x)$ and $k_2(x)$, respectively, and the function $\omega^+(\xi)$, according to Theorem 3, is determined by formulas (13), (15), and (11).

The transposed equation is investigated analogously,

$$\psi(x) + \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{k_1(t-x)}{1 + e^{-t}}\,\psi(t)\,dt + \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{k_2(t-x)}{1 + e^{t}}\,\psi(t)\,dt = h(x),$$

$$-\infty < x < \infty,$$

whose solution can also be written in quadratures.

At present the theory of integral equations of convolution type has been developed quite fully, as may be judged from the handbook ([3]), Ch. VIII, as well as from works in which methods for the approximate solution of these equations are set forth ([8,9]), etc. The theory of equations of smooth transition can undoubtedly be raised to the same level. One may hope that, like "paired" equations and the Wiener–Hopf equation, equations of smooth transition will find a wide range of applications.

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