ON GROUPS WITH INVARIANT NOT QUITE SPLITTABLE SUBGROUPS

MATHEMATICS

1970

SovietRxiv

Abstract

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UDC 519.44

MATHEMATICS

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ON GROUPS WITH INVARIANT NOT QUITE SPLITTABLE SUBGROUPS

(Presented by Academician V. M. Glushkov, 7 I 1970)

1. A subgroup H of a group G is called a θ-subgroup if it has property θ, and a $\bar{\theta}$-subgroup in the opposite case. A number of works of recent years have been devoted to the study of groups in which every $\bar{\theta}$-subgroup is invariant (we shall call such groups $\bar{\theta}J$-groups; see the literature in the survey of S. N. Chernikov (1)), where cyclicity, commutativity, nilpotency, and certain other properties were taken as θ. It is convenient to include among the $\bar{\theta}J$-groups Dedekind groups and, on the other hand, groups in which every proper subgroup is a θ-subgroup. If θ is an absolute group-theoretic property inherited by subgroups of θ-groups, but not conversely, then knowledge of $\bar{\theta}$-groups, all of whose proper subgroups are θ-subgroups, plays an essential role in describing $\bar{\theta}J$-groups. Indeed, in these cases the factor group of a $\bar{\theta}J$-group by its own minimal $\bar{\theta}$-subgroup is Dedekind, so that almost the entire commutant of the group under study turns out to be known.

In the present paper, as θ we take the property of complete splitting of a group.

Definition (2). A group G is called **completely splittable** if it can be represented as the set-theoretic sum of some set of its locally cyclic subgroups, intersecting pairwise in the identity subgroup.

Denoting by r the property of a group of being completely splittable, we arrive at the class of $\bar{r}J$-groups. Naturally, we consider only those $\bar{r}J$-groups which have $\bar{r}$-subgroups, and in what follows this will not be stated separately.

$\bar{r}$-Groups all of whose proper subgroups are r-groups were introduced by P. G. Kontorovich (3) and called z-groups. All locally finite z-groups are finite; their description is given in (3,4). The question of the existence of infinite z-groups is open. It can be proved that an infinite z-group, if it exists, coincides with its commutant. Therefore $\bar{r}J$-groups have been studied under the additional assumption of local finiteness, and in the aperiodic case, of local solubility.

The class of $\bar{r}J$-groups obviously contains all $\bar{c}J$-groups, i.e. groups with invariant noncyclic subgroups. The structure of locally finite $\bar{c}J$-groups is known (5–7);

therefore, in describing $\bar{r}J$-groups we do not present $\bar{c}J$-groups; for brevity, Dedekind groups are regarded as $\bar{c}J$-groups.

Notation: $|g|$ is the order of the element g; $[b,a] = bab^{-1}a^{-1}$; G' is the commutant of the group G; $Z(G)$ is the center of the group G; p, q are distinct primes; $A\lambda B$ is the semidirect product of the groups A and B with invariant factor A and complementary factor B.

2. All finite z-groups are soluble; consequently, all locally finite $\bar{r}J$-groups are soluble. Their structure is completely described by the three theorems below.

Theorem 1. *A finite p-group is a $\bar{r}J$-group if and only if it is of one of the following types:*

1) a $\bar{c}J$-group.

2) $G = G_1 \times A$, A is an elementary abelian p-group (possibly trivial), and $G_1 = \{a_0\}\lambda\{a_1\}\lambda\{a_2\}\lambda \dots \lambda\{a_n\}$, $n \geqslant 2$; $a_0^{p^u} = z$, $u \geqslant 1$, $|z| = |a_1| = \dots = |a_n| = p$; $[a_i, a_{i-1}] = z$, $i = 1, 2, \dots, n$; $[a_i, a_j] = 1$ for $j \leqslant i-2$, $i = 2, 3, \dots, n$.

3) $G = (\{a\} \times \{b\})\lambda\{c\}$, $a^{p^{m-1}} = z$, $m \geqslant 1$, $|z| = |c| = p$, $|b| = p^2$, $[c,a] = 1$, $[c,b] = z$ (in groups of types 1)–3) p is an arbitrary prime number).

4) $G = (\{a\}\lambda\{b\})\lambda\{c\}$, $a^p = z$, $|z| = |b| = |c| = p$, $p > 2$, $[b,a] = z$, $[c,a] = b$, $[c,b] = 1$.

5) $G = (\{a\} \times \{b\})\lambda\{c\}$, $a^p = z$, $|z| = |b| = |c| = p$, $p > 2$, $[c,a] = b$, $[c,b] = z^u$, where $u = 1$ or $u = u_0$, the least quadratic nonresidue modulo p.

6) $G = (\{a\} \times \{b\})\lambda\{c\}$, $a^p = z_1$, $b^p = z_2$, $|z_1| = |z_2| = |c| = p$, $p > 2$, $[c,a] = z_2$, $[c,b] = z_1^u z_2^v$, where $4u + v^2$ is a quadratic nonresidue modulo p.

7) $G = (Q \times \{b\})\lambda\{c\}$, $|b| = 4$, $|c| = 2$, Q is the quaternion group, $[c,g] = 1$ for $g \in Q$, $[c,b] = z$, $z \in Q$, $|z| = 2$.

8) $G = (\{a\}\lambda\{b\})\lambda\{c\}$, $a^4 = z$, $|z| = |b| = |c| = 2$, $[b,a] = z$, $[c,a] = b$, $[c,b] = 1$.

9) $H \subset G \cong G^*$, $G^* = H\lambda C$, $H = \{a\} \times \{b\}$, $|a| = |b| = 4$, $C = \{i\} \times \{j\}$, $|i| = |j| = 2$, and the elements i, j induce on the group H automorphisms defined by the following matrices over the field of four elements:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

10) $G = \{a\}\lambda\{i\}$, $|a| = 8$, $|i| = 2$, $[i,a] = a^2$.

11) $G = (\{a\}\lambda\{i\})\lambda\{j\}$, $|a| = 8$, $a^4 = z$, $|i| = |j| = 2$, $[i, a] = a^{-2}$, $[j, a] = z^{\varepsilon_1}$, $[j, i] = z^{\varepsilon_2}$, $\varepsilon_1, \varepsilon_2 = 0, 1$, $\varepsilon_1 + \varepsilon_2 < 2$.

Theorem 2. A finite nonprimary group G is a $\bar{r}J$-group if and only if it is of one of the following types:

1) a $\bar{c}J$-group.
2) A Frobenius group of the form $C = (\{a\} \times \{b\})\lambda\{g\}$, where $|a| = p$, $|b| = p^2$ or $|b| = pq$.
3) $G = \{a\}\lambda(\{b\}\lambda\{c\})$, $|a| = p > 2$, $|b| = 4$, $|c| = 2$, $bab^{-1} = a^{-1}$, $ca = ac$, $cbc = b^{-1}$.
4) $G = (\{a\} \times \{d\} \times \{b\})\lambda\{c\}$, $|a| = |d| = p > 2$, $|b| = 4$, $|c| = 2$, $cac = a^{-1}$, $cdc = d^{-1}$, $cbc = b^{-1}$.
5) $G = (\{a\}\lambda\{b\}) \times (\{c\}\lambda\{d\})$, $|a| = p$, $|b| = |d| = q$, $|c| = r$, r is a prime, $r \neq q$, the subgroups $\{a\}\lambda\{b\}$ and $\{c\}\lambda\{d\}$ are noncyclic.
6) $G = [(\{a\} \times \{b\})\lambda\{d\}] \times \{c\}$, where $(\{a\} \times \{b\})\lambda\{d\}$ is a Frobenius group, $|a| = |b| = p$, $|d| = |c| = q$.
7) $G = (\{a\}\lambda\{b\}) \times B$, $|a| = p$, $|b| = q$, the subgroup $\{a\}\lambda\{b\}$ is noncyclic, B is a group of exponent q, either abelian of rank $\geqslant 2$, or nonabelian of order q^3.
8) $G = (\{a\} \times \{b\} \times \{c\})\lambda\{d\}$, $|a| = p$, $|b| = |c| = |d| = q$, the subgroup $\{a\}\lambda\{d\}$ is noncyclic, $db = bd$, $dcd^{-1} = bc$.
9) $G = P\lambda\{g\}$, P is a Sylow p-subgroup of exponent p, and:

a) if P is abelian, then $G = \{a\} \times (P_1\lambda\{g\})$, the subgroup P_1 is noncyclic, and for any $g^k \neq 1$ there are no proper g^k-admissible subgroups in P_1;

b) if P is nonabelian, then $P' = z(p) = Z(G) = \{a\}$, and for any $g^k \neq 1$ there are no proper g^k-admissible subgroups in P different from $\{a\}$.

We shall make a number of remarks about the proofs of Theorems 1 and 2.

The following is valid.

Proposition. The commutator subgroup of a finite $\bar{r}J$-group is contained in every one of its z-subgroups.

It follows from the description of finite z-groups, therefore, that the commutator subgroup of a finite primary $\bar{r}J$-group is contained in a subgroup of order p^3 with an element of order p^2. It turns out that for $p \neq 2$ the commutator subgroup may be either cyclic of order p (groups of types 2), 3) of Theorem 1) or abelian of type (p, p) (groups of types 4), 5), 6)). For $p = 2$ there also exist groups (types 9), 10)) with a cyclic commutator subgroup of order 4.

The proof of Theorem 2 is based on consideration of finite nonprimary $\bar{r}J$-groups containing z-subgroups of all possible types, and on the description of finite solvable completely decomposable groups (^{9}).

It follows from Theorems 1 and 2 that an infinite locally finite $\bar{r}J$-group has a simple commutator subgroup.

Theorem 3. *An infinite locally finite group belongs to the class of $\bar{r}J$-groups if and only if it is of one of the following types:*

1) a $\bar{c}J$-group.
2) $G = G_1 \times A$, where A is an elementary abelian p-group, and G_1 is a central product over the subgroup $\{a^p\}$ of some set of isomorphic groups of type $\{a\}\lambda\{b\}$, $|a| = p^{\mu+1}$, $\mu \geq 1$, $|b| = p$, $[b, a] = a^{1+p^\mu}$, and, possibly, a cyclic group of order $p^{\mu+1}$ or a group of type p^∞.
3) $G = (A \times \{b\})\lambda\{c\}$, A is a group of type p^∞, $|b| = p^2$, $|c| = p$, $[c, a] = 1$ for $a \in A$, $[c, b] = z$, $|z| = p$, $z \in A$.
4) $G = (\{a\}\lambda\{b\}) \times B$, $|a| = p$, $|b| = q$, B is an elementary abelian q-group of infinite rank, and $\{a\}\lambda\{b\}$ is noncyclic.

Remark. In groups of type 2) of Theorem 1 and Theorem 3 every nonisolated subgroup is invariant (a subgroup H of a group G is isolated in G if H contains every cyclic subgroup of G with which it has a nontrivial intersection (^{8})). One can show that if a group G has a nonisolated subgroup and all nonisolated subgroups in G are invariant, then G is a group of the type indicated above.

3. **Theorem 4.** *A nonperiodic locally solvable $\bar{r}J$-group contains a mixed abelian subgroup.*

Theorem 4 follows from the following lemma:

Lemma. *If in a torsion-free locally solvable group G all noninvariant subgroups are completely decomposable, then the group G itself is completely decomposable.*

Theorem 5. Let G be a nonabelian group containing a mixed abelian subgroup. G is an $\bar{r}J$-group if and only if its commutator subgroup is a cyclic subgroup of prime order p, and all elements of finite order generate a locally cyclic p-subgroup or the quaternion group.

If, in addition, it is required that G be a group with a finite number of generators, then $G = G_1 \times A$, where A is a free abelian group of finite rank, and the subgroup G_1 is no longer decomposable in a nontrivial way into a direct product. Defining relations have been found for the group G_1.

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Received
30 XII 1969

CITED LITERATURE

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