ON THE APPROXIMATION OF FUNCTIONS SATISFYING A LIPSCHITZ CONDITION BY ALGEBRAIC POLYNOMIALS

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ON THE APPROXIMATION OF FUNCTIONS SATISFYING A LIPSCHITZ CONDITION BY ALGEBRAIC POLYNOMIALS

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The aim of the present article is to obtain estimates for the deviations of functions satisfying a Lipschitz condition from their polynomials of best approximation and from polynomials constructed by the method of least squares.

We shall first prove one general theorem on the approximation of continuous functions by algebraic polynomials.

Let $f(x)$ be a function continuous on the interval $[a, b]$. Let $P_n^f(x)$ be its polynomial of best approximation of degree not exceeding n,

$$\varepsilon_f = \max_{x \in [a,b]} |f(x) - P_n^f(x)| > 0.$$

Let $R_{n,m}^f(x)$ be a polynomial of degree not exceeding n, constructed by the method of least squares with weight function $\omega(x) > 0$ for the function $f(x)$ with nodes at the points

$$hk = a + \frac{b-a}{m-1}k, \quad k = 0, 1, \ldots, m.$$

In what follows, without loss of generality we shall assume that $a = h$ and, consequently, $b = mh$.

Theorem 1. For any integers m and n $(m \geq n+1)$ there exists a number $\theta_{n,m}$ such that, for every function $f(x) \in C$ on the interval $[h, mh]$, the inequality

$$|R_{n,m}^f(x) - P_n^f(x)|/\varepsilon_f \leq \theta_{n,m}$$

holds.

Proof. Denote

$$\varepsilon(kh) = f(kh) - P_n^f(kh), \quad k = 1, 2, \dots, m.$$

To determine the coefficients a_i of the polynomial

$$R_{n,m}^f(x) = \sum_{i=0}^{n} a_i x^i$$

by the least-squares method, we write the system of normal equations

$$\sum_{k=1}^{m} \omega(kh) \sum_{i=0}^{n} a_i k^{i+l} h^i = \sum_{k=1}^{m} \omega(kh) k^l \left[P_n^f(kh) + \varepsilon(kh)\right], \tag{1}$$

where $l = 0, 1, \dots, n$. Denote by $\bar{a}_i$ the coefficients of the polynomial

$$P_n^f(x) = \sum_{i=0}^{n} \bar{a}_i x^i.$$

Substituting the expression for $P_n^f(x)$ into equations (1) and carrying out simple transformations, we obtain

$$\sum_{i=0}^{n} (a_i - \bar{a}_i) h^i \sum_{k=1}^{m} \omega(kh) k^{i+l} = \sum_{k=1}^{m} \omega(kh) k^l \varepsilon(kh), \tag{2}$$

where $l = 0, 1, \dots, n$.

Denote

$$b_i = a_i - \bar{a}_i, \quad s_\alpha = \sum_{k=1}^{m} \omega(kh) k^\alpha.$$

We write out the solution of the system of equations (2)

$$b_i = \frac{1}{h^i} \sum_{k=1}^{m} \omega(kh) \varepsilon(kh) \frac{\Delta_{n,k,i}}{\Delta_n},$$

where

$$\Delta_n = \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ . & . & . & . \\ s_n & s_{n+1} & \dots & s_{2n} \end{vmatrix},$$

and the determinants $\Delta_{n,k,i}$ are obtained from Δ_n by replacing the $(i+1)$-st column by the column

$$\begin{vmatrix} 1 \\ k \\ \dots \\ k^n \end{vmatrix}.$$

Substituting the found values of the coefficients b_i into the identity

$$R^f_{n,m}(x) - P^f_n(x) \equiv \sum_{i=0}^{n} b_i x^i$$

and carrying out simple transformations, we obtain

$$R^f_{n,m}(x) - P^f_n(x) = \sum_{k=1}^{m} \omega(kh)\varepsilon(kh) \sum_{i=0}^{n} \frac{\Delta_{n,k,i}}{\Delta_n} \left(\frac{x}{h}\right)^i.$$

Denote

$$\lambda_{n,k}(x) = \sum_{i=0}^{n} \frac{\Delta_{n,k,i}}{\Delta_n} \left(\frac{x}{h}\right)^i. \tag{3}$$

Then

$$R^f_{n,m}(x) - P^f_n(x) = \sum_{k=1}^{m} \omega(kh)\varepsilon(kh)\lambda_{n,k}(x). \tag{4}$$

It is easy to see that the factors $\lambda_{n,k}(x)$ do not depend on the function $f(x)$. Denote

$$\theta_{n,m} = \max_{x\in[h,mh]} \sum_{k=1}^{m} \omega(kh) \left|\lambda_{n,k}(x)\right|. \tag{5}$$

Then

$$\left|R^f_{n,m}(x) - P^f_n(x)\right| = \left|\sum_{k=1}^{m} \omega(kh)\varepsilon(kh)\lambda_{n,k}(x)\right| \leq$$

$$\leq \varepsilon_f \sum_{k=1}^{m} \omega(kh) \left|\lambda_{n,k}(x)\right| \leq \varepsilon_f \theta_{n,m}.$$

Theorem 2. In the class of continuous functions $f(x)$ satisfying on the interval $[h, mh]$ the Lipschitz condition with constant M, there exists a function $f_0(x)$ such that

$$\max_{x\in[h,mh]} \frac{\left|R_{n,m}^{f_0}(x) - P_n^{f_0}(x)\right|}{\varepsilon_{f_0}} = \theta_{n,m},$$

i.e. the estimate given in Theorem 1 is sharp even in the class of functions $f(x) \in \mathrm{Lip}_M 1$.

First we shall prove the following lemma.

Lemma 1. Let m and n be arbitrary integers $(m \geq n + 1)$ and let α_k, $k = 1, 2, \ldots, m$, be any prescribed sequence of numbers taking the values ± 1. Then for $M > 0$ there exists a function $f(x) \in \mathrm{Lip}_M 1$ on the interval $[h, mh]$ such that

$$\varepsilon(kh) = \alpha_k \varepsilon_f, \qquad k = 1, 2, \ldots, m.$$

Proof of Lemma 1. First define the function $f_1(x)$ so that $f_1(kh) = \alpha_k$, $k = 1, 2, \ldots, m$, where the numbers α_k are those given in the lemma. Choose additional points y_i, $i = 1, 2, \ldots, s$, distinct from kh, so that, putting $f_1(y_i)$ equal to $+1$ or -1, the function $f_1(x)$ changes sign on the total set of points

$$A = \{kh\} \cup \{y_i\}$$

at least $n + 2$ times. Extend the function $f_1(x)$ to the whole interval $[h, mh]$ so that it is continuous on it and linear on each of the partial intervals obtained by partitioning the interval $[h, m]$ by the set A.

The function $f_1(x)$ satisfies a Lipschitz condition with some constant K. Then the function

$$f(x) = \frac{M}{K} f_1(x)$$

satisfies the Lipschitz condition with constant M on the interval $[h, mh]$, $P_n^f(x) \equiv 0$, $\varepsilon_f = M/K$, and therefore

$$\varepsilon_f(kh) = f(kh) - P_n^f(kh) = \alpha_k \varepsilon_f, \qquad k = 1, 2, \ldots, m.$$

Proof of Theorem 2. Let the function

$$\sum_{k=1}^{m} \omega(kh) \left|\lambda_{n,k}(x)\right|$$

attain its greatest value on the interval $[h, mh]$ at the point x_0.

Consider the numerical sequence

$$\alpha_k(x_0) = \mathrm{sign}(\lambda_{n,k}(x)), \qquad k = 1, 2, \ldots, m.$$

Choose a function $f_0(x)$ satisfying Lemma 1, taking as the sequence $\{\alpha_k\}$ the sequence $\{\alpha_k(x_0)\}$. Then for the function $f_0(x)$, in accordance with (4), we have

$$\left|R^f_{n,m}(x_0) - P^f_n(x_0)\right| = \left|\sum_{k=1}^{m} \omega(kh)\varepsilon(kh)\lambda_{n,k}(x_0)\right| =$$

$$= \left|\sum_{k=1}^{m} \omega(kh)\varepsilon_{f_0} \operatorname{sign}(\lambda_{n,k}(x_0))\lambda_{n,k}(x_0)\right| =$$

$$= \varepsilon_{f_0} \sum_{k=1}^{m} \omega(kh) \left|\lambda_{n,k}(x_0)\right| = \varepsilon_{f_0}\theta_{n,m}.$$

Lemma 2. For any integers m and n $(m \geq n+1)$ and an arbitrary weight function $\omega(x) > 0$,

$$\sum_{k=1}^{m} \omega(kh)\lambda_{n,k}(x) = 1.$$

for all x belonging to the interval $[h, mh]$.

Corollary 1. For any integers m and n $(m \geq n+1)$ and an arbitrary weight function $\omega(x) > 0$, $\theta_{n,m} \geq 1$.

Corollary 2. For any natural number m and an arbitrary weight function $\omega(x) > 0$, $\theta_{0,m} = 1$.

Theorem 3. For any integer $m \geq 2$ and the weight function $\omega(x) = 1$,

$$\theta_{1,m} = 5/3 - 4/3(m+\delta);$$

$$\delta = 0 \text{ for } m = 3k-1,\ k = 1, 2, \ldots; \qquad \delta = 1 \text{ for } m = 3 \text{ or } 3k+1,$$
$$k = 1, 2, \ldots$$

Proof. From formulas (5) and (3), for $\omega(x) = 1$ and $n = 1$ we have

$$\theta_{1,m} = \max_{x\in[h,mh]} \sum_{k=1}^{m} \left|\sum_{i=0}^{1} \frac{\Delta_{1,k,i}}{\Delta_1}\left(\frac{x}{h}\right)^i\right| = \max_{x\in[h,mh]} \sum_{k=1}^{m} \left|\frac{(s_0k - s_1)x/h - (s_2 - s_1k)}{s_2s_0 - s_1^2}\right|.$$

Expressing s_0, s_1, and s_2 in terms of m, we obtain

$$\theta_{1,m} = \max_{x\in[h,mh]} \frac{2}{m(m^2-1)} \sum_{k=1}^{m} \left| 3(2k-m-1)\frac{x}{h} + (m+1)(2m+1-3k)\right|.$$

Let us note that each term in the written sum is a function convex downward. Consequently, the whole sum is also a function convex downward, and therefore attains its maximum at one of the endpoints of the interval $[h, mh]$. It is easy to verify that at the endpoints of the interval $[h, mh]$ the values of this function are equal. Hence,

$$\theta_{1,m} = \frac{2}{m(m+1)} \sum_{k=1}^{m} |\, 2(m+1) - 3k \,|.$$

Carrying out the summation, we obtain the expression for $\theta_{1,m}$ given in Theorem 3.

We give several values: $\theta_{2,3} = 1.25, \quad \lim_{m\to\infty} \theta_{2,m} = 1 + 0.48\sqrt{6}$.

We now turn to estimates of the deviations of functions $f(x) \in \mathrm{Lip}_M\, 1$ from their algebraic polynomials constructed by the method of least squares. Consider the inequalities

$$|R^f_{n,m}(x) - f(x)| \le |R^f_{n,m}(x) - P^f_n(x)| + |P^f_n(x) - f(x)| \le \theta_{n,m}\varepsilon_f + \varepsilon_f = \varepsilon_f(\theta_{n,m}+1).$$

It is known that for functions $f(x) \in \mathrm{Lip}_M\, 1$ on the interval $[a, b]$, $\varepsilon_f \le C_n(b-a)M$. An asymptotic estimate of the quantity C_n was obtained by S. M. Nikol' skii in paper (1) and has the form

$$C_n = \frac{\pi}{4n}(1 - \delta_n),$$

where $\delta_n > 0$ and $\delta_n = O(1/\ln n)$ as $n \to \infty$.

The exact values of C_n for $n = 0, 1, 2$ are

$$C_0 = \frac{1}{2}, \qquad C_1 = \frac{1}{4}, \qquad C_2 = 3 - 2\sqrt{2}.$$

Thus, for any function $f(x) \in \mathrm{Lip}_M\, 1$ on the interval $[a, b]$, the inequalities

$$|P^f_n(x) - f(x)| \le C_n(b-a)M,$$

$$|R^f_{n,m}(x) - f(x)| \le C_n(b-a)M(\theta_{n,m} + 1),$$

hold, where the quantities C_n and $\theta_{n,m}$ for $n = 0, 1$, and 2 were given above.

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