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Abstract

Full Text

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On Limit Sets of Functions Harmonic Inside a Sphere

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Let S be the sphere of radius one in three-dimensional Euclidean space R^3. A point $a \in S$ is called an ambiguous point $(1, 4)$ of a function f, defined in the ball $\operatorname{int} S$, if there exist two paths $l_{a;r_1}$ and $l_{a;r_2}$, lying in $\operatorname{int} S$ and ending at this point, along which f has the asymptotic values r_1 and r_2, respectively, where r_1 and r_2 are real numbers distinct from one another.

G. Piranian proved (1) that one can construct a continuous function in the ball $\operatorname{int} S$ for which all points of S are ambiguous points.

The nonexistence of a function in the disk with analogous behavior at the boundary circle follows from an earlier result of F. Bagemihl $(2, 4)$, that any complex-valued function in the disk can have at most a countable set of ambiguous points on the boundary circle. Subsequently F. Bagemihl (3), making essential use of the above-mentioned result of G. Piranian, proved that one can construct a function h, harmonic in $\operatorname{int} S$, such that for every number $r \in R^1$ and every point $a \in S$ there exists a path $l_{a;r} \subset \operatorname{int} S$, ending at the point a, along which h has the asymptotic value r. In these works no estimates are made of the growth rates of the constructed functions near S.

Theorem. *There exists a function H, harmonic in $\operatorname{int} S$, such that for any point $a \in S$ and any interval $[A; B]$ (which may be unbounded both on one side and on the other, or may degenerate into a point when $A = B$) there exists a rectifiable path $l_{a;[A;B]} \subset \operatorname{int} S$, ending at a, orthogonal to S at the point a, along which the limit set (4) of the function H is equal to $[A; B]$, i.e.*

$$C_{l_{a;[A;B]}}(H) = [A; B].$$

Moreover, this function can be constructed in such a way that inside the sphere S_i of radius $1 - \frac{1}{2^i}$, concentric with S, one has $|H(P)| <$

$$< \exp\{\exp\{\ldots \exp \text{const}\}\},$$

where the number of exp' *s is equal to $2i$, and* const *is an absolute constant.*

Remark 1. The indicated result of F. Bagemihl (3) follows from our theorem when $A = B = r$.

Remark 2. The paths $l_{a;[A;B]}$ can be chosen in such a way that $H : l_{a;[A;B]} \to [A;B]$, i.e. the values of H on $l_{a;[A;B]}$ do not go outside the interval $[A;B]$.

Remark 3. The condition of the theorem imposed on the limit set—that it be an interval—is necessary, since the limit set of any real continuous function along any path is an interval.

Remark 4. Let λ be an arbitrary smooth arc tangent to the segment PQ at the point P. Rotating λ about PQ, we obtain a certain conical horn K.

It is asserted that the function H can be constructed in such a way that it will have the limiting behavior specified in the theorem inside any preassigned cone K, i.e., along the indicated paths $l_a; [A;B] \subset \operatorname{int} K_a$, where K_a is the cone K with vertex at the point a and with axis directed along the radius S at this point. However, in this case the restriction on the rate of growth of H indicated in the theorem may fail to hold.

Physical interpretation. If the function H indicated in the theorem is regarded as a stationary solution of the heat-conduction equation (in the general case, as a stationary scalar field), then we arrive at the following physical interpretation of the result obtained.

Suppose that at the center of the ball there is a set of particles $\{M\}$, each of which has a definite heat resistance, determined by an interval $[A;B]$ depending on M: at a temperature within the range $A \le t \le B$ the particle M exists, while for t outside $[A;B]$ the particle M is destroyed.

It is asserted that there exists a stationary thermal regime inside the ball that makes it possible to carry any particle M without destroying it from the center to an arbitrary point on the sphere, in such a way that any other particle M' with another resistance $[A';B']$, which does not withstand at least some temperatures from $[A;B]$, i.e., $[A;B] \setminus [A';B'] \neq \emptyset$, would be destroyed if it followed M along the same path.

At the same time, thermal insulation of the particles is allowed inside any sphere (depending on the particle) concentric with S and lying strictly inside it; thus, when moving toward a point $a \in S$, beginning from some moment M necessarily loses its thermal insulation.

In conclusion, I express my deep gratitude to my scientific adviser, Corresponding Member of the Academy of Sciences of the USSR S. N. Mergelyan, for posing the problem and for valuable guidance.

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Note: Figure translations are in progress. See original paper for figures.