SYMMETRIC SPACES OF MEASURABLE OPERATORS

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Abstract

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SYMMETRIC SPACES OF MEASURABLE OPERATORS

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We consider the space with measure on projections introduced by I. E. Segal (^{1}), $\Gamma = (H, \mathfrak{A}, m)$, i.e., a totality consisting of a Hilbert space H, a Neumann algebra $\mathfrak{A}$ of bounded operators acting in H, and a nonnegative measure m defined on the orthoprojections from $\mathfrak{A}$.

1°. Let A be an operator measurable with respect to the algebra $\mathfrak{A}$; then the operators E_λ ($E_{\lambda-0} \neq E_\lambda$) from the spectral decomposition of the identity for the operator $|A|$ belong to the algebra $\mathfrak{A}$. Denote by $C_0(\Gamma)$ the totality of all operators for which the function $m(I - E_\lambda)$ is finite for all $\lambda > 0$. In this case we shall call the function $n_A(\lambda) = m(I - E_\lambda)$ the distribution function of the operator A. We introduce into consideration the nonincreasing, left-continuous function $s_A(\alpha)$ by the formula

$$s_A(\alpha) = \inf\{\lambda;\ n_A(\lambda) < \alpha\}.$$

The values of the function $s_A(\alpha)$ will be called the s-numbers of the operator A. It follows from the definition that $s_A(\alpha) \to 0$ as $\alpha \to \infty$.

The minimax formula is valid

$$s_A(\alpha) = \inf_{m\mathfrak{N}^\perp < \alpha} \sup_{f \in \mathfrak{N} \cap D(A)} \frac{\|Af\|}{\|f\|}, \tag{1}$$

where the measure of the subspace $\mathfrak{N}^\perp$ is understood as the measure of the corresponding orthoprojector, if it belongs to the algebra $\mathfrak{A}$, and as ∞ otherwise.

For factors of finite type an analogue of formula (1) was established by F. J. Murray and J. von Neumann (see (^{2})). A result of J. Allahverdiev is generalized (see (^{3})): for any operator from $C_0(\Gamma)$

$$s_A(\alpha) = \inf_K \|A - K\|,$$

where K runs through the set of all operators for which the measure $\overline{R(K)}$ is less than α. From the two assertions just given, in the usual way, one derives a number of properties of s-numbers (see (3), Ch. I, § 2).

2°. For a positive operator $A \in C_0(\Gamma)$ one constructs a self-adjoint operator $\widetilde{A} \in n_A(A)$, where $n_A(\lambda)$ is the distribution function of the operator A, extended by zero at zero. If P_α is the spectral decomposition of the identity for the operator $\widetilde{A}$ and $\widetilde{E}_\alpha = P_\alpha = P_{+0}$, then the equality

$$A = \int_0^\infty s_A(\alpha)\, d\widetilde{E}_\alpha, \tag{2}$$

is valid, which we call the Schmidt decomposition of the operator A.

In the set of all measurable operators one singles out the Banach space $\mathcal{L}_1(\Gamma)$ (see (1)) of integrable operators, for which the integral or trace $m(A)$ is defined. Moreover, $\|A\|_{\mathcal{L}_1(\Gamma)} = m(|A|)$ and $|m(A)| \le \|A\|_{\mathcal{L}_1(\Gamma)}$. It turns out that

$$\|A\|_{\mathcal{L}_1(\Gamma)} = \int_0^\infty s_A(\alpha)\, d\alpha. \tag{3}$$

Using formulas (2) and (3), a number of integral inequalities for s-numbers are established.

Let $\psi(\alpha)$ $(0 \le \alpha < \infty)$ be a nondecreasing concave function and $\psi(0) = 0$; then

$$\int_0^\infty s_{A+B}(\alpha)\, d\psi(\alpha) \le \int_0^\infty s_A(\alpha)\, d\psi(\alpha) + \int_0^\infty s_B(\alpha)\, d\psi(\alpha),$$

$$\int_0^\infty s_{A_1 A_2 \dots A_n}(\alpha)\, d\psi(\alpha) \le \int_0^\infty s_{A_1}(\alpha) s_{A_2}(\alpha) \dots s_{A_n}(\alpha)\, d\psi(\alpha).$$

3°. A linear normed space E of measurable operators is called **symmetric** if, for any operator $A \in E$ and measurable operator B, it follows from the relation

$$s_B(\alpha) \le s_A(\alpha) \quad (0 < \alpha < \infty)$$

that $B \in E$ and $\|B\|_E \le \|A\|_E$.

The simplest examples of symmetric spaces are the spaces $\mathcal{L}_p(\Gamma)$ $(1 \le p \le \infty)$.

Theorem 1. *From every sequence of operators in a symmetric space E that converges to zero, one can extract a subsequence that converges to zero almost everywhere.*

There is a general method for constructing symmetric spaces, proposed in the scalar case by E. M. Semenov (see (4)).

Let $\psi_t(\alpha)$ $(t \in T)$ be a family of nondecreasing concave functions on $[0, \infty)$, equal to zero at zero. Consider the family $E_T(\Gamma)$ of all measurable operators for which

$$\|A\|_{E_T(\Gamma)} = \sup_{t \in T} \int_0^\infty s_A(\alpha)\, d\psi_t(\alpha) < \infty. \tag{4}$$

Theorem 2. *The space $E_T(\Gamma)$, with respect to the norm (4), is a Banach symmetric space.*

4°. For symmetric Banach spaces of measurable operators the embedding theorem is valid.

Theorem 3. *If E is a Banach symmetric space, then*

$$\mathcal{L}_1(\Gamma) \cap \mathcal{L}_\infty(\Gamma) \subset E \subset \mathcal{L}_1(\Gamma) + \mathcal{L}_\infty(\Gamma),$$

and moreover

$$\|A\|_{\mathcal{L}_1(\Gamma)+\mathcal{L}_\infty(\Gamma)} = \inf\{\|A_1\|_{\mathcal{L}_1(\Gamma)} + \|A_2\|_{\mathcal{L}_\infty(\Gamma)}\} = \int_0^1 s_A(\alpha)\, d\alpha, \tag{5}$$

where the infimum is taken over all representations $A = A_1 + A_2$ $(A_1 \in \mathcal{L}_1(\Gamma),\ A_2 \in \mathcal{L}_\infty(\Gamma) = \mathfrak{A})$.

Equality (5) is a special case of the following assertion.

Theorem 4. *If $A \in C_0(\Gamma)$, then*

$$\int_0^t s_A(\alpha)\, d\alpha = \inf_{A = A_1 + A_2} \{\|A_1\|_{\mathcal{L}_1(\Gamma)} + t\|A_2\|_{\mathcal{L}_\infty(\Gamma)}\}.$$

This assertion for the scalar case was obtained in ([5],[6]).

5°. Various interpolation theorems are generalized to symmetric spaces of measurable operators (see ([7])).

Theorem 5. *In order that the space E of measurable operators be an interpolation space between the spaces $\mathcal{L}_1(\Gamma)$ and $\mathcal{L}_\infty^0(\Gamma) = \mathcal{L}_\infty(\Gamma) \cap C_0(\Gamma)$, it is necessary and sufficient that, for every $A \in E$, from the relation*

$$\int_0^t s_B(\alpha)\, d\alpha \leq \int_0^t s_A(\alpha)\, d\alpha \quad (t > 0)$$

it follows that $B \in E$.

We note that an analogue of Theorem 5 for the scalar case was obtained by A. P. Calderon ([8]), and for symmetrically normed ideals of completely continuous operators by G. I. Russo ([9]).

From this theorem it follows, in particular, that, up to an equivalent renorming, every interpolation space between $\mathcal{L}_1(\Gamma)$ and $\mathcal{L}^0_\infty(\Gamma)$ is symmetric. Further, it follows from Theorem 5 that the space $E_T(\Gamma)$, which was discussed in Theorem 2, is always an interpolation space between $\mathcal{L}_1(\Gamma)$ and $\mathcal{L}_\infty(\Gamma)$.

Linear mappings acting in spaces of measurable operators will, following (^{3}), be called transformers.

Theorem 6 (M. Riesz convexity theorem). *Every transformer T acting continuously from $\mathcal{L}_{1/\alpha_0}(\Gamma_0)$ to $\mathcal{L}_{1/\beta_0}(\Gamma_1)$ and from $\mathcal{L}_{1/\alpha_1}(\Gamma_0)$ to $\mathcal{L}_{1/\beta_1}(\Gamma_1)$ acts continuously from $\mathcal{L}_{1/\alpha_t}(\Gamma_0)$ to $\mathcal{L}_{1/\beta_t}(\Gamma_1)$, and moreover*

$$\|T\|_{\mathcal{L}_{1/\alpha_t}\to\mathcal{L}_{1/\beta_t}} \le \|T\|^{1-t}_{\mathcal{L}_{1/\alpha_0}\to\mathcal{L}_{1/\beta_0}} \|T\|^{t}_{\mathcal{L}_{1/\alpha_1}\to\mathcal{L}_{1/\beta_1}},$$

where

$$\alpha_t = (1-t)\alpha_0 + t\alpha_1, \qquad \beta_t = (1-t)\beta_0 + t\beta_1 \quad (0 \le t \le 1).$$

6°. The method by which Theorems 5 and 6 were obtained makes it possible to carry over to symmetric spaces of measurable operators also other interpolation theorems proved for symmetric spaces of measurable functions. This method is based on the following construction.

Let $\mathfrak{a}$ be a weakly closed subring of the algebra $\mathfrak{A}$, containing the identity operator. We shall call the subring $\mathfrak{a}$ proper if the aggregate $(H, \mathfrak{a}, m|_\mathfrak{a}) = \gamma$, where $m|_\mathfrak{a}$ is the restriction of the measure m to the orthoprojectors from $\mathfrak{a}$, is a space with measure on the projectors.

For $A \in \mathcal{L}_\infty(\Gamma)$, the expression $m(A \cdot X)$ $(X \in \mathcal{L}_1(\gamma))$ will be a linear continuous functional on $\mathcal{L}_1(\gamma)$ and, consequently, it is representable (see (^{1})) in the form

$$m(A \cdot X) = m(Q(A) \cdot X),$$

where $Q(A) \in \mathcal{L}_\infty(\gamma)$. Thus a projection transformer Q from $\mathcal{L}_\infty(\Gamma)$ to $\mathcal{L}_\infty(\gamma)$ is defined. It can be extended to a continuous transformer mapping $\mathcal{L}_1(\Gamma)+\mathcal{L}_\infty(\Gamma)$ to $\mathcal{L}_1(\gamma) + \mathcal{L}_\infty(\gamma)$.

Theorem 7. *The transformer Q maps every space $E_T(\Gamma)$ into $E_T(\gamma)$ and has norm one.*

The transformer Q makes it possible to pass from transformers acting in the spaces $E_T(\Gamma)$ to transformers acting in the simpler spaces $E_T(\gamma)$, and conversely. By choosing the subrings $\mathfrak{a}$ to be commutative, we establish connections between interpolation theorems for operators acting in spaces of measurable functions and analogous theorems for transformers in spaces of measurable operators.

We note that the indicated method for the case of symmetrically normed ideals of completely continuous operators is close to the method described in the dissertation of M. Z. Solomyak (^{10}).

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Note: Figure translations are in progress. See original paper for figures.

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