STRUCTURE OF EXTREMAL POLYNOMIALS WITH A COMMON SUBDISTRIBUTION

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Abstract

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MATHEMATICS

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STRUCTURE OF EXTREMAL POLYNOMIALS WITH A COMMON SUBDISTRIBUTION

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The article preserves the terminology adopted in our monograph (^{1}).

§ 1. **A polynomial of a prescribed subdistribution.** If $P_n(x)$ is reduced on $[0,1]$, i.e. $\max_{[0,1]} |P_n| = 1$, with full distribution

$$(\overset{\pm}{\sigma}_i)_1^s, \quad [P_n(\overset{+}{\sigma}) = +1;\ P_n(\overset{-}{\sigma}) = -1],$$

then its resolvent

$$R_s(x) = \prod_1^s (x - \sigma_i)$$

is also called the resolvent of the distribution. A simple node of $P_n(x)$ will mean a point $\sigma \in [0,1]$ at which $P_n(\sigma) = \pm 1$, and, if $0 < \sigma < 1$, then $P_n'(\sigma) = 0$, $P_n''(\sigma) \neq 0$; while if $\sigma = 0$ or 1, then $P_n'(\sigma) \neq 0$. In all cases the square (conditional) of the resolvent is called: for interior nodes

$$R_s^2(x) = \prod_1^s (x - \sigma_i)^2;$$

for one endpoint node

$$R_s^2(x) = x \prod_2^s (x - \sigma_i)^2$$

or

$$R_s^2(x) = (1 - x) \prod_1^{s-1} (x - \sigma_i)^2,$$

for two endpoint nodes:

$$R_s^2(x) = x(1 - x) \prod_2^{s-1} (x - \sigma_i)^2.$$

In all cases on $[0,1]$ we have

$$R_s^2(x) \geq 0.$$

Theorem 1. *If $P_n(x)$ is a polynomial of class I or II (^{1}) (reduced!) with simple nodes and with full distribution $(\overset{\pm}{\sigma}_i)_1^s$ ($s \geq 2$ and there is alternation), then there always exist reduced polynomials with the same full distribution of the form*

$$P_n(x) - \alpha R_s^2(x) \tag{1}$$

for sufficiently small $|\alpha|$.

Indeed, (1) gives

$$P_n(\overset{\pm}{\sigma}) - \alpha R_s^2(\overset{\pm}{\sigma}) = \pm 1;$$

it remains to satisfy the reducedness condition. Let $\alpha > 0$; then the condition

$$0 \leq \alpha R_s^2(x) \leq P_n(x) + 1,$$

which is always possible. For $\alpha < 0$ we have

$$P_n(x) - 1 \leq \alpha R_s^2(x) \leq 0,$$

which is also possible.

Corollary. *If $P_n(x)$ is of class II ($n <$ degree of $R_s^2(x)$), then (1) gives the nearest polynomials in degree with distribution $(\overset{\pm}{\sigma}_i)_1^s$. If $P_n(x)$ is of class I, there is always an infinite set of polynomials of degree n with the same $(\overset{\pm}{\sigma}_i)_1^s$.*

Denote by $M_{n,s}$ the set of all reduced polynomials of exactly degree n and lower, containing $(\overset{\pm}{\sigma}_i)_1^s$ as their subdistribution; then between any two elements of $M_{n,s}$ the relation (^{1}) holds

$$Q_n^{(2)}(x) = Q_n^{(1)}(x) - \alpha \varphi_k(x) R_s^2(x), \tag{2}$$

where $\alpha \gtreqless 0$ and $\varphi_k(x) = x^k + \cdots$; $k + 2s \leq n$. In view of (2), $M_{n,s}$ is expressed through any $L_n(x) \in M_{n,s}$:

$$M_{n,s} = \{L_n(x) - \alpha \varphi_k(x) R_s^2(x)\} \tag{3}$$

for all admissible $\varphi_k(x)$. We shall distinguish two cases. A. $M_{n,s}$ contains a unique $L_n(x)$. B. $M_{n,s}$ is a set, always infinite (for example, weighted means). An example of case A is given by the subdistribution

$$T_n(x) = \cos n \arccos(2x - 1)$$

with two neighboring nodes ([1], p. 65); for case B see [1], p. 73.

On the basis of Theorem 1: if $L_n(x)$ is of class I, then we always have case B. Case A is possible only when $L_n(x)$ is of class II (with a subdistribution of class I); but here case B is also possible.

Remark 1. Suppose $M_{n,s}$ has been constructed by formula (3). If $M_{p,s}$ exists for $p > n$ (the polynomials of exact degree p must enter into $M_{p,s}$), then it is always an infinite set, and

$$Q_p^{(2)}(x) = Q_p^{(1)}(x) = \alpha\varphi_k(x)R_s^2(x)$$

$(k = 0, 1, \dots, p - 2s)$, and if $k + 2s < p$, then also

$$Q_p^{(1)}(x) - \alpha\psi_l(x)\varphi_k(x)R_s^2(x) \in M_{p,s} \tag{4}$$

provided that $l + k + 2s \leq p$ and $0 \leq \psi_l \leq 1$ on $(0, 1)$.

Remark 2. If $M_{n,s}$ is a set, then $M_{n+1,s}$ exists and contains polynomials with different signs of the leading coefficients (q_{n+1}). Indeed, in the form (3), where $k + 2s \leq n$, in passing to the form (4) we choose $\psi(x) = x^{n+1-2s-k}$ and $\psi(x) = (1 - x)x^{n-2s-k}$.

Theorem 2. *If, for the constructed $M_{n,s}$, the set $M_{p,s}$ $(p \geq n)$ exists, then it always contains $Q^{(x)}_{p(\max)}(x)$ and $Q^{(x)}_{p(\min)}(x)$—polynomials with $q_p = \max$ and $q_p = \min$, and each of them is unique in $M_{p,s}$.*

Indeed, (q_p) is a bounded set, and $Q_{p(\max)}$, $Q_{p(\min)}$ exist.

Suppose there are two $Q^{(1)}_{p(\max)}(x)$ and $Q^{(2)}_{p(\max)}(x)$; then

$$Q^{(2)}_{p(\max)}(x) = Q^{(1)}_{p(\max)}(x) - \alpha\varphi_k(x)R_s^2(x),$$

where $k + 2s < p$; in the case $\operatorname{sgn}(-\alpha) = \operatorname{sgn} q_{p(\max)}$ we take, according to the scheme (4), $\psi(x) = x^{p-k-2s}$; in the case $\operatorname{sgn}\alpha = \operatorname{sgn} q_{p(\max)}$, we take $\psi(x) = (1 - x)x^{p-k-2s-1}$. In both cases we obtain a polynomial with $q_p > q_{p(\max)}$.

Remark 3. Neither $q_{p(\max)}$ nor $q_{p(\min)}$ is equal to zero for $p > n$ (see Remark 2). But also for $p = n$: suppose $q_{n(\min)} = 0$; then $M_{n,s}$ is lowered, i.e., there is $M_{n-1,s} \subset M_{n,s}$; and then in $M_{n,s}$ it is necessary that $q_{n(\max)}$ and $q_{n(\min)}$ have different signs.

Corollary 1. *A necessary and sufficient condition for the nonlowerability of $M_{n,s}$ is that in it $\{q_n\}$ is strictly sign-constant.*

Corollary 2. *If $M_{n,s}$ is a set, then $Q_{n(\max)}(x)$ and $Q_{n(\min)}(x)$ are always of class II, since, if $L_n(x) \in M_{n,s}$ and is of class I, then, according to the form (1), its leading coefficient can give neither a max nor a min.*

§ 2. Indices of subdistributions

Suppose a complete distribution $L_n(x)$ contains p nodes; from it a subdistribution of class I

$$(\sigma_i^{\pm})_1^s$$

is chosen; let

$$(\sigma_i^{\pm})_1^{s'}$$

be the remaining subdistribution ($p = s + s'$). Then

$$M_{n,s} = \{L_n(x) - \alpha\varphi_k(x)R_s^2(x)\}; \tag{5}$$

here $k + 2s \leq n$.

Definition. The **index** of $L_n(x)$ relative to $(\sigma_i^{\pm})_1^s$ is the integer nonnegative number $k^*_{(s)}$ equal to the least possible degree of $\varphi_k(x)$ in the scheme (5). This number is, obviously, always unique.

Corollary 1. Since the reducedness of the polynomials in scheme (5) requires that

$$L_n(x) - 1 \leq \alpha\varphi_k(x)R_s^2(x) \leq L_n(x) + 1,$$

the curve $\alpha\varphi_k(x)R_s^2(x)$ passes through, with tangency (one-sided or two-sided), all the points $(\bar{\sigma}_i^{\pm})_1^s$, while the points $(\sigma_i'^{\pm})_1^{s'}$ must be passed through by means of $\varphi_k(x)$. Thus, $k^*_{(s)}$ is the minimum number of roots on $[0, 1]$ necessary for this. Consequently, we have

$$\varphi_k(x) = \prod_1^{k_s^*}(x - \lambda_i).$$

Corollary 2. If in scheme (5) it turns out that $k_s^* > n - 2s$, then $\alpha = 0$ and $M_{n,s} \equiv L_n(x)$ (in this case $L_n(x)$ is always of class II), and conversely. Thus, the necessary and sufficient condition for the existence of the set $M_{n,s}$ is

$$0 \leq k_s^* \leq n - 2s.$$

Corollary 3. If $M_{n,s}$ is a set, then it always contains "fully impoverished" polynomials, i.e. such polynomials whose complete distribution is

$$(\sigma_i^{\pm})_1^s.$$

Indeed, let us form the subset

$$\left\{ L_n(x) - \alpha\prod_1^{k^*}(x - \lambda_i)R_s^2(x) \right\} \subset M_{n,s}, \tag{6}$$

where $k^*_{(s)} \le n-2s$ and the (λ_i) are all distinct in the admissible intervals on $[0,1]$. After choosing the (λ_i), for sufficiently small $|\alpha|$ we obtain fully impoverished polynomials.

Corollary 4. If $L_n(x)$ has index $k^*_{(s)} = n-2s$, then the entire set $M_{n,s}$ is expressed in the form

$$M_{n,s} = \left\{ L_n(x) - \alpha \prod_1^{n-2s} (x-\lambda_i) R_s^2(x) \right\} \tag{7}$$

with a unique representation for each $P_n(x) \in M_{n,s}$.

Theorem 3. If $L_n(x)$ has index $k^*_{(s)} (\le n-2s)$, then in every subset of the form

$$\left\{ L_n(x) - \alpha \prod_1^{k^*_s} (x-\lambda_i) R_s^2(x) \right\} = \widetilde{M}_{n,s} \tag{8}$$

under all values of (λ_i) and α admissible by the reducedness conditions, the number α remains sign-constant.

Let $\widetilde{M}_{n,s}$ contain $P_n^{(1)}(x)$ and $P_n^{(2)}(x)$ with the corresponding $(\lambda_i), \alpha_0$ and $(\lambda'_i), \alpha'_0$. Put $\alpha_0 > 0$, $\alpha'_0 < 0$; for all $0 < \alpha < \alpha_0$ and $\alpha'_0 < \alpha' < 0$ the reducedness is preserved. Then the polynomial

$$L_n(x) - \frac{1}{2}\left[\alpha \prod (x-\lambda_i) + \alpha' \prod (x-\lambda'_i)\right] R_s^2(x)$$

belongs to $M_{n,s}$, and, choosing $\alpha' = -\alpha$, we obtain polynomials of the form

$$L_n(x) - \beta \varphi_{k^*_s - 1}(x) R_s^2(x) \in \widetilde{M}_{n,s},$$

which contradicts the condition.

Corollary. If $k^*_s = n-2s$, then $\widetilde{M}_{n,s} \equiv M_{n,s}$ and, consequently, $L_n(x)$ is one of the extreme polynomials; for $\alpha > 0$, $L_n \equiv Q_{n(\max)}$, for $\alpha < 0$, $L_n \equiv Q_{n(\min)}$. Thus, the necessary and sufficient condition that $L_n(x)$ be extreme is: its index $k^*_{(s)} = n-2s$.

Sufficiency has been proved; necessity follows from the form

$$M_{n,s} = \left\{ Q_{n(\max)}(x) - \alpha \prod_1^{k^*_s} (x-\lambda_i) R_s^2(x) \right\},$$

where, by virtue of the uniqueness of the polynomial with $q_{n(\max)}$, it is necessary that $k^*_s = n-2s$ and $\alpha > 0$ throughout $M_{n,s}$. The case $Q_{n(\min)}(x)$ is analogous.

Theorem 4. In $M_{n,s}$, both extreme polynomials $Q_{n(\max)}(x)$ and $Q_{n(\min)}(x)$ have $(\overset{\pm}{\sigma}_i)_1^s$ as their common greatest subdistribution.

Consider the unique representation

$$Q_{n(\min)}(x) = Q_{n(\max)}(x) - \alpha^* \prod_1^{n-2s} (x - \lambda_i^*) R_s^2(x); \qquad (9)$$

here $\alpha^* = \max$. Suppose that, in addition to $(\overset{\pm}{\sigma}_i)_1^s$, the extreme polynomials have at least one more common node, for definiteness $\overset{+}{\sigma}$. Then formula (9) will take the form (after a change of numbering)

$$Q_{n(\min)}(x) = Q_{n(\max)}(x) - \alpha^* \prod_1^{n-2s-2} (x - \lambda_i^*) R_s^2(x)(x-\sigma)^2.$$

We shall prove that here α^* can be increased by replacing $(x-\sigma)^2$ by $(x-\lambda')(x-\lambda'')$. Indeed, if, for example, we set $\lambda' = \sigma - \varepsilon$ and $\lambda'' = \sigma + \varepsilon$, then $(x-\lambda')(x-\lambda'') = (x-\sigma)^2 - \varepsilon^2$. Further, since the curve

$$\alpha^* \prod_1^{n-2s-2} (x - \lambda_i^*) R_s^2(x)(x-\sigma)^2$$

must lie within the bounds $Q_{n(\max)}(x) \pm 1$, with some points of tangency, then upon replacing it by

$$\alpha^* \prod_1^{n-2s-2} (x - \lambda_i^*) R_s^2(x)(x-\lambda')(x-\lambda''),$$

this curve remains within the same bounds for sufficiently small ε, no longer having points of tangency with the bounds. Consequently, α^* can be increased, which is impossible.

The results obtained answer the questions of uniqueness and non-reducibility of the solution in the problem of V. A. Markov (^{1}). This problem is equivalent to finding an extremal polynomial of a not absolutely monotone segment-functional $(\mu_k)_0^n$. Replacing the segment by $\mu_0, \mu_1, \ldots, \mu_{n-1}, \theta$, we have, for any $-\infty < \theta < +\infty$, a segment of class II except perhaps for only one point $\theta = \theta^0$. Thus, only at θ^0 can the solution of the problem fail to be unique and can it be reducible. This point, as well as the true distribution $(\overset{\pm}{\sigma}_i)_1^s$ of the corresponding segment, are found by an algebraic method $((^1), p.62)$.

The principal extremal polynomial $Q_N(x)$ for such a distribution is constructed by the methods indicated in $((^1), p.21)$. By the very meaning of the problem,

$N \leq n$. If $N < n$, the solution is reducible, and the question of uniqueness is determined by the index of $Q_N(x)$ relative to $(\overset{\pm}{\sigma}_i)_1^s$; for $k_s^* > n-2s$, the solution (of degree not exceeding n) is unique; for $k_s^* \leq n-2s$, there exists $M_{n,s}$, the set of solutions of the problem. If $N = n$, then for $k_s^* > n-2s$ we have both uniqueness and non-reducibility; for $k_s^* \leq n-2s$, there exists a non-reducible set of solutions $M_{n,s}$, and the point θ^0 has been called singular by us $((^1), p.72)$.

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[1] E. V. Voronovskaya, *The Method of Functionals and Its Applications*, L., 1963.

Note: Figure translations are in progress. See original paper for figures.

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