ON A PURSUIT PROBLEM

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Abstract

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In the present paper, which is related to the investigations ($^{1-14}$), sufficient conditions are discussed under which the approximating extremal strategy $U_a^{(e)}$ guarantees to the pursuer a convergence with the pursued at the time ϑ, or no later than the time ϑ.

Let the pursuing ($y[t]$) and the pursued ($z[t]$) motions be described by the equations

$$\dot{y} = f^{(1)}(t, y, u), \qquad \dot{z} = f^{(2)}(t, z, v), \tag{1}$$

where y and z are n-dimensional phase vectors of the objects; u and v are r-dimensional control vectors constrained by the condition

$$u \in \mathcal{U}, \qquad v \in \mathcal{V}, \tag{2}$$

the sets $\mathcal{U}$ and $\mathcal{V}$ being bounded and closed, and $f^{(i)}$ continuous functions satisfying Lipschitz conditions with respect to x. We shall call any absolutely continuous function $y[t]$ or $z[t]$ on the interval $[t_0, \vartheta]$ satisfying the contingency (^{15})

$$\dot{y}[t] \in \mathcal{F}^{(1)}(t, y[t]) \quad \text{or} \quad \dot{z}[t] \in \mathcal{F}^{(2)}(t, z[t])$$

for almost all $t \in [t_0, \vartheta]$ a motion $y[t]$ or $z[t]$. Here $\mathcal{F}^{(i)}(t, q)$ denotes the convex hulls of the sets swept out by the vectors $f^{(i)}(t, q, w)$, when w ranges over $\mathcal{U}$ or $\mathcal{V}$, respectively. Motions satisfying the initial conditions $y[t_*] = y_*$ or $z[t_*] = z_*$ will be denoted by the symbols $y[t, t_*, y_*]$ or $z[t, t_*, z_*]$. A meeting of the motions $y[t]$ and $z[t]$ is determined by the condition $(z[t] - y[t]) \in \mathcal{L}$, where $\mathcal{L}$ is a given closed set. Introduce the $2n$-dimensional vector $x = \{y, z\}$. Denote by the letter $\mathcal{M}$ the set in the space $\{x\}$ specified by the condition $(z - y) \in \mathcal{L}$. By the symbol $\rho(x, \mathcal{M})$ we denote the distance from x to $\mathcal{M}$.

We shall say that, for a fixed initial position $\{t_0, x_0\} = \{t_0, y_0, z_0\}$, the approximating strategy (^{14}) U_a guarantees convergence of the motions $y[t]$ and $z[t]$ at the time $\vartheta > t_0$, if

$$\lim_{\delta\to 0}\sup\left[\sup_{z[t],\,y_\delta[t]}\rho(x[\vartheta],\mathcal{M})\right]=0. \tag{3}$$

Here $z[t]$ is an arbitrary motion $z[t,t_0,z_0]$; $y_\delta[t]$ is the motion $y_\delta[t,t_0,y_0]$ generated by the piecewise-constant control $u_\delta[t]$ assigned by the strategy U_a. If it is possible to construct a system of sets $\mathcal{W}(t)$ $(t_0 \leq t \leq \vartheta)$, strongly u-stable $(^{14})$ and satisfying the conditions $\mathcal{W}(\vartheta)=\mathcal{M}$, $x_0 \in \mathcal{W}(t_0)$, then the extremal $(^{14})$ strategy $U_a^{(e)}$ toward them, according to $(^{14})$, will guarantee the convergence of $y[t,t_0,y_0]$ and $z[t,t_0,z_0]$ at the time ϑ. (In our case closed sets $\mathcal{W}(t)$, by definition, are strongly u-stable if the following condition is fulfilled: whatever $t_* \in [t_0,\vartheta)$, $x_* = \{y_*,z_*\} \in \mathcal{W}(t_*)$, and $\delta \in (0,\vartheta-t_*]$ may be, for every motion $z(t,t_*,z_*)$ one can choose a motion $y(t,t_*,y_*)$ so that the inclusion $x(t_*+\delta,t_*,x_*) \in \mathcal{W}(t_*+\delta)$ holds.)

Let $\mathcal{W}^*(t,\vartheta)$ $(t \leq \vartheta)$ be the set of all x for which the process $x(t)$ absorbs the programmed set $\mathcal{M}$ at the time ϑ from the position $\{t,x\}$ $(^{14})$. As the sets $\mathcal{W}(t)$ it is convenient to choose the sets $\mathcal{W}^*(t,\vartheta)$ provided they are strongly u-stable. We shall therefore indicate some sufficient conditions for strong u-stability of the sets $\mathcal{W}^*(t,\vartheta)$.

Let $x_* = \{y_*,z_*\} \in \mathfrak{W}^*(t_*,\vartheta)$. Choose $t^* > t_*$ $(t^* < \vartheta)$ and define $\mathcal{Y}(x_*)$ as the totality of all points $y(t^*,t_*,y_*)$ which are obtained by ranging over all possible motions $y(t,t_*,y_*)$. The set $\mathcal{Y}(x_*)$ is bounded and closed. Fix some motion $z^0(t,t_*,z_*)$ $(t_* \leq t \leq t^*)$. Denote $\hat{z}^0 = z^0(t^*,t_*,z_*)$. Let the point $x^* = \{y^*,\hat{z}^0\} \notin \mathfrak{W}^*(t^*,\vartheta)$. Then one can single out a certain set $\mathcal{Z}_{[t_*,\vartheta]}(x^*)$ of motions $z(t,t_*,z_*)$ which for $t_* \leq t \leq t^*$ coincide with the motion $z^0(t,t_*,z_*)$ and at the same time have the property that for each of them it is impossible to choose a motion $y(t,t^*,y^*)$ effecting an encounter $(z(\vartheta)-y(\vartheta)) \in \mathcal{L}$. Further, for each $z(t,t_*,z_*) \in \mathcal{Z}_{[t_*,\vartheta]}(x^*)$ one can choose some set $\mathcal{Y}_{[t_*,\vartheta]}(z(t))$ of motions $y(t,t_*,y_*)$ effecting an encounter $(z(\vartheta)-y(\vartheta)) \in \mathcal{L}$. The totality of all motions $y(t,t_*,y_*) \in \mathcal{Y}_{[t_*,\vartheta]}(z(t))$, corresponding to all $z(t) \in \mathcal{Z}_{[t_*,\vartheta]}(x^*)$, determines a certain set of points $y = y(t^*,t_*,y_*)$. We denote this set by the symbol $\mathcal{Y}(x^*)$. Obviously, $\mathcal{Y}(x^*) \subset \mathcal{Y}(x_*)$.

Lemma 1. *If, for all sufficiently small* $\delta = t^* - t_* > 0$, *for all possible points* x_* *the sets* $\mathcal{Y}(x_*)$ *are convex, and if all sets* $\mathcal{Y}(x^*)$ *can be chosen closed, convex, and upper semicontinuous with respect to inclusion under variation of* x^*, *then the sets* $\mathfrak{W}^*(t,\vartheta)$ *are strongly* u*-stable.*

Suppose, to the contrary, that the lemma is false. Then for some point $x_* = \{y_*,z_*\} \in \mathfrak{W}(t_*,\vartheta)$ one can fix a motion $z^0(t,t_*,z_*)$ such that all points $x^* = \{y,z^0(t^*,t_*,z_*)\}$ with $y \in \mathcal{Y}(x_*)$ will lie outside $\mathfrak{W}^*(t^*,\vartheta)$. But then, in accordance with the preceding, one can construct a mapping of the points $y^* \in \mathcal{Y}(x_*)$ to the sets $\mathcal{Y}(x^*) \subset \mathcal{Y}(x_*)$. According to theorem $(^{16})$, in this case there is found a point y^0 satisfying the condition $y^0 \in \mathcal{Y}(x^0)$, where $x^0 = \{y^0,z^0\}$. But by the construction of the sets $\mathcal{Y}(x^*)$ this is impossible, since it follows that at $t = t^*$

there passes through the point x^0 a motion $x(t, t_*, x_*)$ which both arrives at $\mathcal{M}$ at the instant ϑ, and at the same time in no way can arrive at $\mathcal{M}$ at this instant. The contradiction proves the lemma.

The choice of the sets $\mathcal{Y}(x^*)$ which appear in the lemma is stipulated with a certain degree of arbitrariness. Making Lemma 1 more concrete, one can choose as the set $\mathcal{Z}_{[t_*,\vartheta]}(x^*)$ the totality of all motions $z(t, t^*, z^0)$ solving the problem for the program maximin

$$\max_{z(t,t^*,z^0)} \min_{y(t,t_*,y^*)} \rho(x(\vartheta), \mathcal{M}),$$

and as $\mathcal{Y}_{[t_*,\vartheta]}(z(t))$ choose all motions $y(t, t_*, y_*)$ effecting an encounter $(z(\vartheta) - y(\vartheta)) \in \mathcal{L}$. Then the sets $\mathcal{Y}(x^*)$ will necessarily be closed and upper semicontinuous with respect to inclusion (in x^*). In this case the conditions of Lemma 1 reduce only to the requirement of convexity of $\mathcal{Y}(x_*)$ and $\mathcal{Y}(x^*)$. Finally, in the case where system (1) is linear and the sets $\mathcal{U}, \mathcal{V}$, and $\mathcal{L}$ are convex, the conditions of Lemma 1 are replaced by a single condition of convexity of the set $\{v(t)\}$ of all those controls $v(t)$ $(t^* \leq t \leq \vartheta)$ which solve the problem for the program maximin

$$\max_v \min_u \rho(x(\vartheta, t^*, x^*), \mathcal{M}).$$

This condition is obviously fulfilled if the condition of uniqueness of the vector l^0, indicated in papers (^{13,14}), is fulfilled. Thus, the conditions of Lemma 1 in this particular case reduce to the already known conditions of strong u-stability of the sets $\mathfrak{W}^*(t, \vartheta)$.

We shall say that, for a fixed initial position $\{t_0, x_0\} = \{t_0, y_0, z_0\}$, an approximating strategy U_a guarantees the **approach** of the motions $y[t]$ and $z[t]$ by the instant $\vartheta > t_0$, if

$$\lim_{\delta \to 0} \sup \left[\sup_{z[t],\, y_\delta[t]\; t_0 \leq t \leq \vartheta} (\inf \rho(x[t], \mathcal{M})) \right] = 0. \tag{4}$$

If it is possible to construct a system of sets $\mathfrak{W}(t)$ $(t_0 \leq t \leq \vartheta)$ satisfying the conditions $\mathfrak{W}(\vartheta) = \mathcal{M}$, $\mathcal{M} \subset \mathfrak{W}(t)$, $x_0 \in \mathfrak{W}(t_0)$ and u-stable (^{14}), then, according to (^{14}), the extremal strategy $U_\varepsilon^{(e)}$ for them will guarantee the approach of $y[t, t_0, y_0]$ and $z[t, t_0, z_0]$ by the instant ϑ. Let $\mathfrak{W}^0(t, \vartheta)$ $(t \leq \vartheta)$ be the set of all $x = \{y, z\}$ for which the process $x(t)$ falls-

gives the program set $\mathcal{M}$ at the moment ϑ from the position $\{t, x\}$ (^{14}). As the sets $\mathcal{W}(t)$ it is convenient to choose the sets $\mathcal{W}^0(t, \vartheta)$ $(t_0 \leq t \leq \vartheta)$, under the condition of their u-stability. We indicate some sufficient conditions for the u-stability of the sets $\mathcal{W}^0(t, \vartheta)$. For this purpose one should repeat the construction of the sets $\mathcal{U}(x_*)$ and $\mathcal{U}(x^*)$ described above, but here with the sole difference that the sets $\mathcal{W}^*$ are replaced by $\mathcal{W}^0$, and the meeting condition $(z(\vartheta) - y(\vartheta)) \in \mathcal{L}$ is replaced by the condition $\min_t \rho(x(t), \mathcal{M}) = 0$ (for $t^* \leq t \leq \vartheta$

in the first case and for $t_* \leq t \leq \vartheta$ in the second). Analogously to Lemma 1, the following assertion is now proved.

Lemma 2. *If, for all sufficiently small $\delta = t^* - t_* > 0$, for all possible points x_* the sets $\mathcal{U}(x_*)$ are convex, and if all the sets $\mathcal{U}(x^*)$ can be chosen closed, convex, and upper semicontinuous with respect to inclusion (in x^*), then the sets $\mathcal{W}^0(t, \vartheta)$ are u-stable.*

Making Lemma 2 more concrete, one may take as the set $\mathcal{Z}_{[t_*,\vartheta]}(x^*)$ the collection of all motions $z(t, t^*, z^0)$ that solve the problem of the program maximum

$$\max_{z(t,t^*,z^0)} \min_{y(t,t^*,y^*)} \min_t \rho(x(t), \mathcal{M})$$

($t^* \leq t \leq \vartheta$), and as $\mathcal{U}_{[t_*,\vartheta]}(z(t))$ take all motions $y(t, t_*, y_*)$ that realize a meeting $(z(t) - y(t)) \in \mathcal{L}$ for $t_* \leq t \leq \vartheta$. Then the conditions of Lemma 2 reduce only to the requirement of convexity of $\mathcal{U}(x_*)$ and $\mathcal{U}(x^*)$.

A consequence of the results from paper (^{14}) and Lemmas 1 and 2 is the following assertion:

Theorem. *Suppose that, for all sufficiently small $\delta = t^* - t_* > 0$, for all possible points x_* the sets $\mathcal{U}(x_*)$ are convex. If, moreover, for the sets $\mathcal{U}(x^*)$ constructed on the basis of $\mathcal{W}^*(t, \vartheta)$ or $\mathcal{W}^0(t, \vartheta)$ ($t_0 \leq t \leq \vartheta$) can be chosen closed, convex, and upper semicontinuous with respect to inclusion (in x^*), then, under the condition $x_0 \in \mathcal{W}^*(t_0, \vartheta)$ or $x_0 \in \mathcal{W}^0(t_0, \vartheta)$, the strategy $U_a^{(\Theta)}$, extremal to the sets $\mathcal{W}^*(t, \vartheta)$ or $\mathcal{W}^0(t, \vartheta)$, guarantees the convergence of the motions $y[t]$ and $z[t]$ at the moment ϑ or by the moment ϑ, respectively.*

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Note: Figure translations are in progress. See original paper for figures.

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