JAMES CLASSES OF MINIMAL SYSTEMS AND THEIR CONNECTION WITH ISOMETRIC PROPERTIES OF B-SPACES

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Abstract

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JAMES CLASSES OF MINIMAL SYSTEMS AND THEIR CONNECTION WITH ISOMETRIC PROPERTIES OF B-SPACES

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1. In the present paper the theory of bases in Banach spaces (B-spaces), due to R. James ([1,2]; see also ([3]), Ch. 4, § 3), is extended to minimal (i.e., biorthogonal) sequences (see § 2). This makes it possible to obtain certain general properties of B-spaces (Theorem 4, Corollaries 1, 2). In § 4 we use the language of β- and δ-modules (introduced in ([4])) to establish connections between the isometric properties of a space (i.e., its unit sphere $S(B) = \{x \in B : \|x\| = 1\}$) and various classes of minimal systems, which are linear-topological objects. Thus minimal systems appear here as the connecting link between the isometric and topological properties of B-spaces. In the final formulations (Theorem 6) they no longer appear. Results of this kind were noted in ([5]).

Let $X = \{x_k\}_1^\infty \subset B$ be a minimal sequence. The sequence of functionals conjugate (i.e., biorthogonal) to X will be denoted by $X^* = \{x_k^*\}_1^\infty \subset B^*$. Thus $x_k^*(x_j) = \delta_{kj}$. A system
$Y = \{y_k\}_1^\infty \subset B$ is called block with respect to X if

$$\left\{ y_k = \sum_{j=n_k+1}^{n_{k+1}} a_{jx} j \right\}_{k=1}^{\infty},$$

where $n_{k+1} > n_k$ $(k = 1, 2, ...)$. We denote by $E(M)$ the closed linear span of the set $M \subset B$. When speaking of subspaces, we shall have in mind only infinite-dimensional subspaces.

A basis $X = \{x_k\}_1^\infty \subset S(B)$ is called **boundedly complete** (introduced in ([6])) if from the estimate

$$\sup_n \left\| \sum_1^n a_{kx} k \right\| < \infty$$

there follows the convergence of the series

$$\sum_{1}^{\infty} a_{kx} k,$$

and it is called **shrinking** ([1]) if the conjugate system X^* is complete in B^*, i.e., $E(X^*) = B^*$. In the case of basic sequences X, i.e., bases in their closed linear span $E(X)$, the use of the indicated terminology presupposes passage to the subspace $E(X)$.

Alaoglu ([7]) showed that bounded completeness of a basis $X \subset B$ implies that $B \simeq E(X^*)^*$, i.e., B is isomorphic to the space conjugate to $E(X^*)$. James ([1]) established that the reflexivity of a space B with basis X is equivalent to the property of the basis of being simultaneously boundedly complete and shrinking, and showed the duality of these properties. Various properties of bases that are not boundedly complete or shrinking, as well as block-bases with respect to such systems, can be found in ([8]).

2. A minimal sequence $X = \{x_k\}_1^\infty \subset B$ will be called **shrinking** if $X^* \subset E(X)^*$ is complete in $E(X)^*$. A minimal system X with total conjugate system X^* on $E(X)$ will be called **boundedly complete**,

if for every bounded sequence $\{y_j\}_{j=1}^\infty \subset E(X)$, from the convergence $f(y_j) \to$ for every $f \in E(X^*)$ there follows the existence of $y_0 \in E(X)$ for which $f(y_j) \to f(y_0)$ $\forall f \in E(X^*)$, which we shall denote by $y_j \to y_0(E(X^*))$.

Remark 1. If B is separable and isomorphic to a conjugate space, $B \simeq B_1^*$, then every complete minimal system X in B with complete conjugate $X^* \subset B_1$ in B_1 is boundedly complete.

Proposition 1. a) *If a boundedly complete minimal system is a basis, then this basis is boundedly complete, and conversely.*

b) *Every block system of a boundedly complete (respectively, spanning) minimal system is boundedly complete (respectively, spanning).*

Corollary 1. *If B is embedded in a separable space isomorphic to a conjugate space, then for every $\varepsilon > 0$ in B there exists $B_\varepsilon \subset B$ ($\dim B_\varepsilon = \infty$) with a boundedly complete basis, ε-isometric* to a conjugate space.*

Corollary 2. *If B has a separable conjugate, then in B there exists a subspace with a spanning basis.*

Theorem 1.** *If in B there exists a complete minimal boundedly complete sequence X, then B is isomorphic to a conjugate space.*

Proof. Let $X^* \subset B^*$ be the system conjugate to X. Denote by QB the natural embedding of B in B^{**}, and $B_0 = E(X^*)^\perp \subset B^{**}$, where, as usual, $E^\perp$ is the annihilator of the space E. In view of the totality of X^* on B, $B_0 \cap QB = 0$. We shall show that $B_0 + QB = B^{**}$. Indeed, let $z \in B^{**}$. By the separability of $E(X^*)$, there exists a bounded sequence $\{z_n\}_1^\infty \subset QB$ such that $z_n \to z(E(X^*))$.

By the definition of bounded completeness of X, it follows that there exists $x_0 \in QB$ such that $z_n \to x_0(E(X^*))$. But then $z - x_0 = y_0 \in E(X^*)^\perp$. Thus, $z = x_0 + y_0 \in QB + B_0$. Hence, $(E(X^*))^\perp + QB = B^{**}$. This means that $(E(X^*))^* = B^{**}/(E(X^*))^\perp \simeq QB = B$.

Theorem 2. *Let X be a complete minimal system in B with a total conjugate. For reflexivity of B it is necessary and sufficient that the system X be simultaneously boundedly complete and spanning.*

The necessity is obvious. The proof of sufficiency is carried out in the same way as the proof of Theorem 1. From the bounded completeness of the minimal system X it follows that $B^{**} = QB + E(X^*)^\perp$ (see the proof of Theorem 1). However, in view of the spanning property of X, the space $E(X^*) = B^*$, and hence $E(X^*)^\perp = 0$. Thus, $B^{**} = QB$, which means reflexivity.

Theorem 3. *The conjugate system to a boundedly complete one is spanning, and conversely.*

Proof. Let X be a boundedly complete minimal sequence. From Theorem 1 it follows that the conjugate system X^* spans $B_1 (= E(X^*))$, where $B_1^* \simeq E(X)$. Hence, $(X^*)^* = X$ is complete in $E(X) \simeq B_1^*$.

Conversely: the conjugate system X to a spanning minimal system X is boundedly complete, since $E(X^*) = E(X)^*$ (see Remark 1).

From the results given in this section it follows easily:

Theorem 4. *If B^{**} is separable, then B and B^* contain infinite-dimensional reflexive subspaces.*

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* B_1 and B_2 are called ε-isometric if there exists an isomorphism $T : B_1 \to B_2$ such that $\|T\| \cdot \|T^{-1}\| \leq 1 + \varepsilon$.

** Analogously, (7) showed that weak completeness (in the topological sense, i.e., on generalized sequences) of B relative to a norming subspace $E \subset B^*$, i.e. such that $\sup\{|f(x)| : f \in S(E)\} \geq c\|x\|$ for some $c > 0$ and all $x \in B$, entails the isomorphism of B and E^*. From the proof of Theorem 1 it is clear that the same follows under only the totality of E, and also from sequential weak completeness relative to a separable $E \subset B^*$.

We also note the following reformulation of Theorem 4: *if B has no reflexive (infinite-dimensional) subspaces, then B^{**} is nonseparable.*

3. We present some properties of minimal systems connected with the notions of normingness and bounded completeness. First we indicate an internal characteristic of normingness of minimal systems, analogous to James' s characteristic ([1]) of norming bases.

Proposition 2. Let $X = \{x_k\}_1^\infty \subset B$ be a minimal system and let the biorthogonal $X^* \subset E(X)^*$. Denote

$$p_n(f) = \sup\{|f(x)| :\ x \in S(E\{x_k\}_{k=n}^{\infty})\}.$$

For completeness of X^* in $E(X)^*$, i.e. for normingness of X, it is necessary and sufficient that $p_n(f) \to 0$ $(n \to \infty)$ for every $f \in B^*$.

Proposition 3. If a monotone basis* $X = \{x_k\}_1^\infty \subset S(B)$ is not boundedly complete, then for every $\theta > 0$ there exists a sequence $\{y_k\}_1^\infty$, block with respect to X, $\|y_k\| = 1$, such that

$$1 - \theta \le \left\|\sum_1^n y_k\right\| \le 1 + \theta$$

for all $n = 1, 2, ...$

Proposition 4. If the basis X is not norming, then for every $\theta > 0$ there exists a sequence $Y = \{y_k\}_1^\infty \subset S(B)$, block with respect to X, such that

$$\left\|\sum_1^\infty \alpha_k y_k\right\| \ge (1-\theta)\sum_1^\infty \alpha_k$$

for all $\{\alpha_k \ge 0\}_1^\infty$.

4. Let $\mathfrak{B}$ be some family of subspaces of the space B. Denote

$$\beta(\varepsilon; x, \mathfrak{B}) = \sup_{E\in\mathfrak{B}} \inf_{y\in S(E)} \|x/\|x\| + \varepsilon y\| - 1$$

and

$$\delta(\varepsilon; x, \mathfrak{B}) = \inf_{E\in\mathfrak{B}} \sup_{y\in S(E)} \|x/\|x\| + \varepsilon y\| - 1.$$

In the case when $\mathfrak{B}$ is the family of all subspaces of finite defect, we shall write $\beta^0(\varepsilon; x, B)$ and $\delta^0(\varepsilon; x, B)$. The values of the characteristics β and δ (β- and δ-moduli) for some concrete spaces are given in ([4]). Let $X = \{x_k\}_1^\infty \subset B$. Denote $\mathfrak{B}(X) = \{E(\{x_k\}_n^\infty)\}_{n=1}^\infty$.

Theorem 5. Let X be a minimal system with total adjoint.

a) If $\beta(\varepsilon_0; x, \mathfrak{B}(X)) = 0$ for some $\varepsilon_0 > 0$, then X is not boundedly complete.

b) If $\delta(\varepsilon; x, \mathfrak{B}(X)) = \varepsilon$ (for all $x \in S(B)$), then $E(X^*) \ne B^*$, i.e. X is not a norming sequence.

The proof of Theorem 5 makes essential use of the results of item 2. Let us outline, for example, the plan of the proof of part a) of the theorem. Starting from the definition of the β-modulus, for a given $\theta > 0$ we construct a sequence $Y = \{y_i\}_{i=0}^\infty \subset S(B)$, block with respect to X, such that

$$1 + \theta \ge \left\|y_0 + \varepsilon_0 \sum_1^n y_k\right\| \ge 1 - \theta$$

for all $n = 1, 2, \ldots$. The obtained inequalities mean that the minimal system Y is not boundedly complete. Since Y is a block system with respect to X, it follows from Proposition 1 that X also is not boundedly complete.

From Theorem 5 and the results of item 2 it follows:

Theorem 6. Let B be a separable Banach space.

a) If $\beta^0(\varepsilon_0; x, B) = 0$ for some $\varepsilon_0 > 0$, then B is not isomorphic to a conjugate space and is not embedded as a subspace in a separable space isomorphic to a conjugate space.

b) If $\delta^0(\varepsilon; x, B) = \varepsilon$, then B^* is nonseparable.

Corollary 3 (I. M. Gelfand ([9])). The space $L_1[0, 1]$ is not iso-

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* A basis $\{x_k\}_1^\infty$ is called θ-monotone if

$$\left\|\sum_1^m a_k x_k\right\| \geq (1-\theta)\left\|\sum_1^n a_k x_k\right\|$$

for all $m \geq n$ and any numbers $\{a_k\}_1^\infty$, and monotone if in the indicated inequality one may take $\theta = 0$.

isomorphic to a subspace of any separable B-space isomorphic to a conjugate one, since $\beta^0(1, x, L_1) \equiv 0$.

Corollary 4 (C. Bessaga and A. Pełczyński ([10])). If a separable space contains a subspace isomorphic to c_0, then B is not isomorphic to a conjugate space, since $\beta^0(1, x, c_0) \equiv 0$.

In connection with item a) of Theorem 6, we note that from the condition $\beta^0(\varepsilon_0, x_0, B) = 0$ ($\varepsilon_0 > 0$) it does not follow that $x_0 \in S(B)$ is not an extreme point of the unit sphere. Thus, for example, $\beta^0(\varepsilon; x(t), C[0, 1]) = 0$ for $0 \leq \varepsilon < 2$ for all elements $x(t) \in S(C)$, including the function $x(t) \equiv 1$.

Theorem 6 (item a)) admits the following strengthening:

Theorem 6′. Let B be a separable space. If for every compact set $K \subset S(B)$ there is an $\varepsilon(K) > 0$ such that $\beta^0(\varepsilon(K); x, B) \equiv 0$ for all $x \in K$, then B is not embedded as a subspace in a separable space isomorphic to a conjugate one.

Theorems 5 and 6 partially reverse the results of items 4 and 5 in ([5]), which we supplement with the following proposition:

Theorem 7. Let B be separable. For every $\theta > 0$ there exists a subspace $B_1 \subset B$ with a θ-monotone basis $X = \{x_k\}_1^\infty \subset S(B_1)$, for which, for every $x \in S(B_1)$, $\beta^0(\varepsilon; x, B_1) \equiv \beta^0(\varepsilon; x, B)$, $\delta^0(\varepsilon; x, B_1) = \delta^0(\varepsilon; x, B)$.

Moreover, if $\mathfrak{B}(X) = \{E(\{x_k\}_{k=n}^\infty)\}_{n=1}^\infty$, then for every $x \in S(B)$ (and not only $x \in S(B_1)$) $\beta(\varepsilon; x, \mathfrak{B}(X)) \equiv \beta^0(\varepsilon; x, B)$ and $\delta(\varepsilon; x, \mathfrak{B}(X)) \equiv \delta^0(\varepsilon; x, B)$.

Here:

a) If $\beta^0(1, x, B) \geq a > 0$, then X is a boundedly complete basis, and hence B_1 is θ-isometric to a conjugate space.

b) If $\delta^0(1, x, B) \leq b < 1$, then X is a shrinking basis, and hence B_1 has a separable conjugate.

c) If simultaneously $\beta^0(1, x, B) \geq a > 0$ and $\delta^0(1, x, B) \leq b < 1$, then B_1 is reflexive.

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Note: Figure translations are in progress. See original paper for figures.