Reports of the Academy of Sciences of the USSR

Astronomy

1970

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1970. Volume 195, No. 4

UDC 521.21

Astronomy

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On the Calculation of the Gravitational Moment J_6 of Jupiter and Saturn

(Presented by Academician V. G. Fesenkov on 29 IV 1970)

The external gravitational potential of a rotating equilibrium planet can be written in the form ([1])

$$V(r,t) = \frac{GM}{r}\left\{1 - \sum_{n=1}^{\infty}\left(\frac{a_1}{r}\right)^{2n} J_{2n}P_{2n}(t)\right\}, \qquad (1)$$

where G is the gravitational constant; M and a_1 are the mass and equatorial radius of the planet; $t = \cos\vartheta$; $P_{2n}(t)$ are Legendre polynomials. The coefficients J_{2n} are called the relative gravitational moments of the planet and are defined by the relation

$$J_n = -\frac{1}{Ma_1^n}\int \rho(r',t')(r')^n P_n(t')\,d\tau. \qquad (2)$$

Here $\rho(r,t)$ is the density distribution in the planet, and the integration is performed over the entire volume of the planet where $\rho \neq 0$.

Until now, for Jupiter and Saturn only the first two moments, J_2 and J_4, have been calculated theoretically and determined from observations of the natural satellites. In works ([1],[2]), a theory was developed for the figure of hydrostatically equilibrium rotating planets with allowance for terms of the third approximation. With the aid of this theory, using the available model density distributions in the planets Jupiter and Saturn, one can theoretically predict the magnitude of the moment J_6 of these planets.

As is seen from (2), the moments J_n are completely determined by the angular dependence of the density $\rho(r,t)$, i.e., by the shape of the internal level surfaces and the external surface of the planet. According to ([1]), the equation of these surfaces can be written in the form

$$r(a,t) = a\left[1 - e\cos^2\vartheta - ({}^3\!/_8 e^2 + k)\sin^2 2\vartheta + {}^1\!/_4\,({}^1\!/_2 e^3 + h)\,(1 - 5\sin^2\vartheta)\sin^2 2\vartheta\right], \tag{3}$$

where a is the equatorial radius of the level surface; $e(a)$, $k(a)$, $h(a)$ are parameters characterizing its shape. For slowly rotating planets e, k, h have, respectively, first, second, and third orders of smallness and are determined from the system of equations ([1])

$$eD - {}^1\!/_2 m - {}^3\!/_5(S+T) = 0,$$

$$(3e^2 - 8k)D - 6eS + 3P + {}^8\!/_3 Q = 0, \tag{4}$$

$$(e^3 + {}^{32}\!/_7 ek - {}^{10}\!/_7 h)D - {}^3\!/_7(e^2+8k)S - {}^{15}\!/_7 eP + {}^{11}\!/_7 H + {}^{32}\!/_{21} eQ + {}^{30}\!/_{91} J = 0,$$

where

$$D(\beta) = \beta^{-3}\int_0^\beta \delta(t)\,dt^3, \qquad dt^3 = \frac{d[t^3]}{dt}\,dt,$$

$$S(\beta) = \beta^{-5}\int_0^\beta \delta(t)\,d[et^5], \qquad P(\beta) = \beta^{-7}\int_0^\beta \delta(t)\,d\left[t^7\left(e^2 + \frac{8}{9}k\right)\right],$$

$$H(\beta) = \beta^{-9}\int_0^\beta \delta(t)\,d\left[t^9\left(e^3 + \frac{192}{143}ek + \frac{30}{143}h\right)\right],$$

$$T(\beta) = \int_\beta^1 \delta(t)\,d[e], \qquad Q(\beta) = \beta^2\int_\beta^1 \delta(t)\,d[t^{-2}k], \tag{5}$$

$$J(\beta) = \beta^4\int_\beta^1 \delta(t)\,[t^{-4}(h - 4ek)].$$

Here $\beta = s/s_1$ is the mean relative radius of the surface; the subscript 1 means that the corresponding quantity refers to the outer surface; $\delta(s) = \rho(s)/\rho_0$ is the relative density; ρ_0 is the mean density of the planet; $m = 3\omega^2/4\pi G\rho_0$ is the small parameter of the theory, and ω is the angular velocity of rotation of the planet.

In practical calculations it is convenient to transform the quantities S, P, H, T, Q, J by integration by parts, transferring the differentiation sign to the density, for example

$$S(\beta) = \delta(\beta)e(\beta) - \beta^{-5}\int_0^\beta t^5 e\,\frac{d\delta}{dt}\,dt. \tag{6}$$

Knowing the density distribution in the planet $\rho(s)$ and solving the system of equations (4), (5) with the aid of electronic computers, one can find the parameters e, k, h as functions of the relative radius β. Then the moments J_2, J_4, J_6 can be calculated. In this case formula (2), which determines the moments J_n, is conveniently written in the form of algebraic relations ([3])

$$J_2 = \frac{2}{3}e_1 - \frac{1}{3}m - \frac{1}{3}e_1^2 + \frac{2}{21}me_1 + \frac{8}{21}k_1,$$

$$J_4 = -\frac{4}{5}e_1^2 + \frac{4}{7}e_1 m - \frac{32}{35}k_1 + \frac{4}{5}e_1^3 - \frac{22}{49}e_1^2 m + \\ + \frac{3616}{2695}e_1 k_1 + \frac{208}{385}mk_1 - \frac{122}{385}h_1, \tag{7}$$

$$J_6 = \frac{8}{7}e_1^3 - \frac{20}{21}e_1^2 m + \frac{128}{77}e_1 k_1 - \frac{160}{231}k_1 m + \frac{80}{231}h_1$$

or in the form of integral relations

$$J_2 = \frac{2}{5}\left(1 - \frac{2}{3}e_1 - \frac{1}{9}e_1^2 - \frac{16}{15}k_1\right)S(1),$$

$$J_4 = -\frac{12}{35}\left(1 - \frac{4}{3}e_1\right)P(1), \qquad J_6 = \frac{8}{21}H(1). \tag{8}$$

It can be shown that the solution of the equations of the theory of figures (4) can be represented in the form of convergent series for power-law density distributions. A special place is occupied by the simplest laws—the linear and quadratic ones,

$$\delta(\beta) = 4(1-\beta), \tag{9}$$

$$\delta(\beta) = \frac{5}{2}(1-\beta^2), \tag{10}$$

which approximate well the real density distribution in hydrogen–helium planets. In writing (9) and (10) it has been taken into account that, for the planets under consideration, $\delta_1 = 0$, and the condition of total mass is $D(1) = 1$. For law (10) the solution is as follows:

$$e = \sum_{i=0}^{\infty} \alpha_i \beta^{2i}; \qquad k = \sum_{i=0}^{\infty} \gamma_i \beta^{2i+2}; \qquad h = \sum_{i=0}^{\infty} \nu_i \beta^{2i+2}. \tag{11}$$

The numerical values of the first several coefficients in these series are equal to

$$\alpha_i/\alpha_0 = 1;\ 0.172;\ 0.0744;\ 0.0378;\ 0.0205;\ 0.0115;\ \dots; \qquad \alpha_0 = 0.619\, m;$$
$$\gamma_i/\alpha_0^2 = 0.041;\ 0.023;\ 0.017;\ 0.013;\ 0.010;\ \dots;$$
$$\nu_i/\alpha_0^3 = 0.166;\ 0.136;\ 0.109;\ 0.089;\ 0.071;\ 0.056;\ \dots. \tag{12}$$

We are interested in the values of the parameters e, k, h at the surface, $\beta = 1$. Series (11) show that e_1, k_1, and h_1, and consequently the shape of the outer surface and the moments J_n, are substantially influenced by the mass distribution throughout the entire planet, and not only in its outer layers. For law (9) the solution is more complicated:

$$e = \sum_{i=0}^{\infty} \alpha_i \beta^{i}; \qquad k = \sum_{i=0}^{\infty} \gamma_i \beta^{i+1} + \ln\beta \sum_{i=0}^{\infty} \bar{\gamma}_i \beta^{i+2};$$
$$h = \sum_{i=0}^{\infty} \nu_i \beta^{i+1} + \ln\beta \sum_{i=0}^{\infty} \bar{\nu}_i \beta^{i+2} + \ln^2\beta \sum_{i=0}^{\infty} \mu_i \beta^{i+4}. \tag{13}$$

The results of calculations for the quadratic density law (10) are given in Table 1.

Table 1

	m	e_1	k_1	h_1	J_2	$-J_4$	J_6	J	K
Jupiter	0.084	0.069	$3.7 \cdot 10^{-4}$	$11.4 \cdot 10^{-5}$	0.0172	$7.6 \cdot 10^{-4}$	$5.3 \cdot 10^{-5}$	0.026	0.0029
Saturn	0.14	0.115	$10.2 \cdot 10^{-4}$	$53.2 \cdot 10^{-5}$	0.028	$2.0 \cdot 10^{-3}$	$2.5 \cdot 10^{-4}$	0.042	0.0074

For the linear density law (9), J_2, J_4 were calculated. The results, together with the corresponding experimental data ([4]), are summarized in Table 2.

Table 2

	e_1	k_1	J	K	$J_{\exp}$	$K_{\exp}$
Jupiter	0.065	$4.1 \cdot 10^{-4}$	0.023	$2.3 \cdot 10^{-3}$	0.02206 ± 0.00022	0.0025 ± 0.0014

	e_1	k_1	J	K	J_{exp}	K_{exp}
Saturn	0.108	$11.6 \cdot 10^{-4}$	0.038	$6.5 \cdot 10^{-3}$	0.02501 ± 0.00003	0.0039 ± 0.0003

The estimates given have an accuracy of order 10% within the adopted density laws. For Jupiter, both density laws (10) and (9) lead to good agreement with the experimental J and K, with the linear law giving even better agreement. It may be assumed that the calculated value $J_6 = 5.3 \cdot 10^{-5}$ has an error of no more than 20% and thus can already be introduced into the theory of motion of Jupiter' s fifth satellite (Amalthea) ([5]).

For Saturn, density laws (10) and (9) give values of J and K that differ from the observed ones, although the discrepancy itself is not so large (less than a factor of two). For Saturn the linear law (9) gives noticeably better results than the quadratic law (10), since it effectively takes into account the concentration of density toward the planet' s center. This shows that the value of the gravitational moments depends substantially on the density distribution throughout the entire planet, including the density distribution in the central regions. The conclusion that the density distribution throughout the whole planet is important for calculating J_n, at first glance, contradicts the intuitive idea that only the density distribution in the outer layers of the planet is significant (the high powers of the radius in the integral for J_n (2)).

In fact, the density distribution over the entire interval $0 \leq \beta \leq 1$ substantially affects the shape of the level surfaces and the intermedi-

···of this on J_n. At present it is not clear whether previous investigators paid sufficient attention to this circumstance. The data of Table 1 and Table 2 can also be interpreted to mean that Saturn has a noticeable concentration of matter toward the center (a large heavy core), and the value $J_6 = 2.5 \cdot 10^{-4}$ given in Table 1 is overestimated by approximately a factor of two.

We take this opportunity to thank O. V. Ivanova for assistance with the numerical calculations.

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Received
18 IV 1970

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Note: Figure translations are in progress. See original paper for figures.