THE BIRTH OF PERIODIC MOTIONS FROM A STATE OF EQUILIBRIUM

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THE BIRTH OF PERIODIC MOTIONS FROM A STATE OF EQUILIBRIUM

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In the present note we consider conditions for the appearance of limit cycles from the system

$$\dot{x} = X(x), \tag{A}$$

which has an equilibrium state with two purely imaginary roots (the remaining ones having nonzero real parts) and a Lyapunov quantity $g_{2m+1} \neq 0$, in passing to a δ-close, up to rank $2m+4$ (see (^{1})), system

$$\dot{x} = X(x) = P(x) = \widetilde{X}(x). \tag{$\widetilde{A}$}$$

A relation is established between the number of generated limit cycles and the order of the Lyapunov quantity g_{2m+1}; a complete classification of the trajectories of system ($\widetilde{A}$) is given. Here x is a vector; $X(x), P(x)$ are vector-functions belonging to the class C^N ($N \geq 2m+4$) (to the analytic class).

In (^{1}) conditions for the appearance of limit cycles from an equilibrium state of a two-dimensional system were studied; in (^{2}, ^{3}), conditions for the birth of one periodic motion from a system having two purely imaginary roots and n negative roots, $g_3 \neq 0$. The general case has not hitherto been considered by anyone. The study is carried out by the method of point mappings; for its construction the system is reduced to a special form, and the neighborhood is split into a noncritical region, in which there are no closed trajectories, and a critical one.

§ 1. **Lemma 1.** *The system ($\widetilde{A}$), by a nondegenerate change of variables and time, can be reduced to the form*

$$\begin{aligned}\dot{x} &= -y + g_{2m+1}x(x^2+y^2)^m + X(x,y,z,w) + P(x,y,z,w),\\ \dot{y} &= x + g_{2m+1}y(x^2+y^2)^m + Y(x,y,z,w) + Q(x,y,z,w),\\ \dot{z} &= A^- z + Z(x,y,z,w) + L(x,y,z,w),\\ \dot{w} &= A^+ w + W(x,y,z,w) + M(x,y,z,w),\end{aligned} \tag{1}$$

where x, y *are scalars;* z, w *are vectors;* A^-, A^+ *are matrices whose characteristic roots have negative and, respectively, positive real part.* X, Y, Z, W, P, Q, L, M *belong to the class* C^{N-1} *(to the analytic class);* X, Y, W *vanish for* $x = 0,\ y = 0,\ w = 0$*;* $X(x,y,0,0),\ Y(x,y,0,0),\ Z(x,y,0,0),\ W(x,y,0,0)$ *vanish for* $x = y = 0$ *together with their derivatives up to order* $2m+1$*;* X, Y, Z, W *are obtained from the right-hand sides of system* (A)*, while* P, Q, L, M *are obtained from the additions.*

For the proof, the ideas of the monographs (^{4}, ^{5}) are used.

§ 2. **Lemma 2** *(on an invariant surface). Suppose that in some neighborhood* $G_x \times G_y$ *of the fixed point* $x = 0,\ y = 0$*, where* x, y *are vectors, a point mapping is given*

$$\bar{x} = Ax + X(x,y), \qquad \bar{Y} = By + Y(x,y), \tag{2}$$

where the spectral radius of the constant matrix B*,* $\|B\|_{\mathrm{sp}} = r_2$*, matrices*

$$A^{-1}\|A^{-1}\|_{\mathrm{sp}}^{-1} = r_1, \quad r_1 > r_2, \quad r_1 > 1; \quad X(x,y),\ Y(x,y)$$

are vector functions that vanish, together with their first-order derivatives, at $x = 0$, $y = 0$, and belong to the class C^k, $k \geq 1$.

Then, in some neighborhood of the origin, there exists an invariant surface $y = \varphi(x)$ belonging to the class C^{k-1}, $\varphi(0) = 0$ ($\varphi'_x(0) = 0$ for $k > 1$), satisfying the Poincaré functional equation

$$\varphi[Ax + X(x,\varphi(x))] = B\varphi(x) + Y(x,\varphi(x)). \tag{3}$$

Lemma 3. Suppose that $\det A \neq 0$, $\|A\|_{\mathrm{sp}} = r_1$, $\|B^{-1}\|_{\mathrm{sp}}^{-1} = r_2$, $r_1 < r_2$, $r_1 < 1$.

Then the mapping (2) has an invariant surface $x = \psi(y)$ (uniqueness may fail), $\psi(0) = 0$, $\psi(y) \in C^{N_0 - 1}$, where $1 \leq N_0 \leq k$.

Lemmas 2 and 3 are proved by constructing a metric space of surfaces and then applying the contraction mapping principle.* On the basis of Lemma 3, one establishes the existence of smooth invariant surfaces $E_1 : w = \psi(r,z)$, $\psi(0,0) = 0$, and $E_2 : z = \chi(r,w)$, $\chi(0,0) = 0$, of dimensions $p+1$ and $q+1$, respectively, defined in some neighborhood of the fixed point of the point mapping

$$\begin{aligned}\bar{r} &= r_0 + f_1(r_0,z_0,w_0),\\ \bar{z} &= s^- z_0 + f_2(r_0,z_0,w_0),\\ \bar{w} &= s^+ w_0 + f_3(r_0,z_0,w_0),\end{aligned} \tag{T}$$

where $r_0, \bar{r}$ are scalars; $z_0, \bar{z}$ are p-dimensional vectors; $w_0, \bar{w}$ are q-dimensional vectors; s^- is a constant $(p \times p)$ matrix with spectrum inside the unit circle, $\det s^- \neq 0$; s^+ is a constant $(q \times q)$ matrix with spectrum outside the unit circle; f_1, f_2, f_3 are nonlinearities belonging to the smoothness class C^{N-1}. Considering the mapping (T) on one of the surfaces of the form $w = \psi(r, z)$ and then selecting the invariant surface $z = \psi_1(r)$, $\psi_1(0) = 0$, we obtain

$$\bar{r} = r_0 + (G_k + f_4(r_0))r_0^k, \quad k = 2, \dots, N-1, \tag{4}$$

where $f_4(0) = 0$, and suppose $G_k \neq 0$ is a quantity which we shall call Lyapunov. An invariant manifold containing the fixed point O will be called incoming (outgoing) if T^{nM} tends asymptotically to O as $n \to \infty$ $(n \to -\infty)$. Denote by $C^1_{i,j}$, $i + j = p + q + 1$, a saddle fixed point with an i-dimensional incoming and a j-dimensional outgoing manifold. Then the following holds.

Lemma 4. 1) Suppose $G_k > 0$, k odd; then we have a saddle $C^1_{p,q+1}$; 2) $G_k < 0$, k odd; we have a saddle $C^1_{p+1,q}$; 3) k even; then the fixed point has a $(p+1)$-dimensional incoming manifold with boundary and a $(q+1)$-dimensional outgoing manifold with boundary.

We shall denote such a saddle point by $C^1_{p+1/2,q+1/2}$, and in what follows we shall regard it as the merging of the saddles $C^1_{p+1,q}$ and $C^1_{p,q+1}$. Trajectories (T^{nM} for increasing and decreasing n) that do not belong to the incoming and outgoing invariant manifolds will be saddle trajectories (cf. (^{11})).

To construct the point mapping in the critical region

$$|w|^2 \leq \sigma r^2, \quad \text{where } r^2 = x^2 + y^2, \quad \sigma > 0, \quad |w|^2 = w_1^2 + \dots + w_q^2, \tag{5}$$

where w_i are the components of the vector w, in equations (1) we make the substitution

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z, \quad w = r\bar{w} \tag{6}$$

in the region (5), $|w|^2 \leq \sigma r^2$, which leads to regular expressions.

* Invariant surfaces of point mappings were studied in ($^{6-8}$). On invariant surfaces of differential equations, see (^{10}).

The mapping of the surface $\varphi = 0$ onto the surface $\varphi = 2\pi$ will take the form

$$\begin{aligned} \bar{r} &= sr_0 + f_1(r_0, z_0, w_0), \\ \bar{z} &= s^- z_0 + f_2(r_0, z_0, w_0), \\ \bar{w} &= s^+ w_0 + f_3(r_0, z_0, w_0). \end{aligned} \tag{7}$$

Here, when the additions are absent, $s = 1$; $s^- = \exp(2\pi A^-)$ is a $(p \times p)$-matrix, $s^+ = \exp(2\pi A^+)$ is a q-dimensional matrix whose characteristic roots for s^- lie inside, and for s^+ outside, the unit circle; $\det s^- \neq 0$. The fixed points of the mapping (7) (with the exception of the trivial one $r_0 = 0$, $z_0 = 0$, $w_0 = 0$, corresponding to the equilibrium state) correspond to limit cycles of equation (1). Reducing the problem on fixed points of the mapping (7) to the problem of determining the roots r_0^* of a defining scalar equation, we see that there will be no more than m positive roots if $g_{2m+1} \neq 0$ is a Lyapunov quantity.

Lemma 5. *In some neighborhood, independent of the particular choice of a system* $(\tilde{A})$ *that is* δ*-close up to order* $2m+4$ *to the system* (A)*, and of the point mapping* (T)*, there exists an invariant surface of the form* $w = \psi(r, z)$*, passing through all fixed points of the mapping.*

Denote by $C_{i,j}$ a cycle in the $(p+q+2)$-dimensional space, $p+q+2 = i+j-1$, which is the intersection of the i-dimensional incoming and the j-dimensional outgoing manifolds. Under the point mapping, the cycle C_{ij} corresponds to a saddle fixed point $C^1_{i-1,j-1}$. Restricting ourselves to cycles of integer multiplicity, we obtain the main result.

Theorem 1. *Let a system* (A) *of class* C^N *(of analytic class), with two purely imaginary roots,* p *roots with negative real part, and* q *roots with positive real part, have the Lyapunov quantity* $g_{2m+1} \neq 0$ $(2m+4 \leq N)$.

Then for a system $(\tilde{A})$ *of class* C^N *(of analytic class) the following holds: a) there exist* $\varepsilon_0 > 0,\ \delta_0 > 0$ *such that, whatever system* $(\tilde{A})$*,* δ*-close up to order* $2m+4$*, we take, in the* ε_0*-neighborhood of* O *(the equilibrium state) there will be no more than* m *closed trajectories; b) for any* $\varepsilon < \varepsilon_0$ *and* $\delta < \delta_0$ *one can specify a system* $(\tilde{A})$ *for which, in the* ε*-neighborhood of the point* O*, there will lie a prescribed number* k $(0 \leq k \leq m)$ *of limit cycles; c) the cycles* $C_{p+2,q+1}$ *and* $C_{p+1,q+2}$ *will alternate (will be adjacent in* r_0*); adjacent to the equilibrium state* $O_{p+2,q}$ *will be the cycle* $C_{p+1,q+2}$*, and adjacent to* $O_{p,q+2}$ *will be the cycle* $C_{p+2,q+1}$*; d) a two-dimensional manifold whose trajectories, as* $t \to \infty$*, tend to one limit cycle* $C_{p+2,q+1}(O_{p+2,q})$*, and as* $t \to -\infty$ *tend to the cycle* $C_{p+1,q+2}(O_{p,q+2})$*, is the intersection of the incoming* $(p+2)$*-dimensional manifold of one cycle and the* $(q+2)$*-dimensional manifold of the other cycle (or equilibrium state); e) all other trajectories not belonging to invariant incoming and outgoing manifolds pass at a finite distance from the generated cycles (and the equilibrium state).*

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CITED LITERATURE

1. A. A. Andronov, E. A. Leontovich, *Matem. sborn.*, **40**, (82), 179 (1956).

2. Yu. I. Neimark, *Radiofizika*, **1**, No. 1, 41 (1958); **1**, No. 2, 95 (1958); **1**, No. 5–6, 146 (1958); *DAN*, **129**, No. 4, 736 (1959).

3. N. N. Brushlinskaya, *DAN*, **139**, No. 1, 9 (1961).

4. A. M. Lyapunov, *The General Problem of the Stability of Motion*, Collected Works, Vol. 2, Acad. Sci. USSR Press, 1956.

5. I. G. Malkin, *Theory of Stability of Motion*, Moscow, 1966.

6. J. Hadamard, *Bull. Soc. Math. France*, **29**, 224 (1901).

7. S. Lattes, *Ann. mat.*, **13**, ser. 3, 1 (1907).

8. D. V. Anosov, *Scientific Reports of Higher School, Physical-Mathematical Sciences*, **3**, No. 1, 3 (1959).

9. Yu. I. Neimark, *Radiofizika*, **10**, No. 3, 311 (1967).

10. A. Kelley, *J. Diff. Equat.*, **3**, No. 4, 546 (1967).

11. R. M. Mintz, *DAN*, **147**, No. 1, 31 (1962).

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