NONDISCRETE TOPOLOGIZABILITY OF INFINITE COMMUTATIVE RINGS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 519.48

MATHEMATICS

V. I. ARNAUTOV

NONDISCRETE TOPOLOGIZABILITY OF INFINITE COMMUTATIVE RINGS

(Presented by Academician P. S. Aleksandrov on 17 III 1970)

It is obvious that in any ring one can introduce a topology in a trivial way (the discrete topology). In finite rings this is the only possible topology. The question of the possibility of introducing a nondiscrete topology in certain infinite rings was considered in papers ([1,3,4,6]). In the present paper it is proved that every infinite commutative ring admits a nondiscrete topologization.

Theorem 1. If a commutative ring R has an infinite nil-ideal, then R has an infinite ideal I such that $I^2 = 0$.

Lemma 1. If a primary group G for the prime number p contains only a finite number of elements of order p, then G is a finite direct sum of finite cyclic groups and groups of type p^∞.

Lemma 2. If a commutative ring R has an infinite ideal I such that $I^2 = 0$ and $p^k I = 0$ for some power k of the prime number p, then R admits a nondiscrete topologization in which I is an open ideal.

Proof. By the first theorem of Prüfer (see ([5]), p. 146), I as a group decomposes into a direct sum of cyclic subgroups G_α, i.e.

$$I = \sum_{\alpha \in \Omega} G_\alpha.$$

If S is a finite subset of Ω, then put

$$N_S = \sum_{\alpha \in S} G_\alpha.$$

For any finite subsets $S \subset \Omega$ and $M \subset R$, define the set

$$V_{M,S} = \{a \in N_S \mid aM \subseteq N_S\}.$$

It is easy to verify that the family of subsets $V_{M,S}$ can be taken as a base of neighborhoods of zero, in order to turn R into a nondiscrete topological ring.

Lemma 3. If a commutative ring R has an infinite ideal I such that $I^2 = 0$ and I, as a group, is a direct sum of a finite number of groups of type p^∞ for some prime number p, then R admits a nondiscrete topologization in which I is an open ideal.

Proof. Let

$$I = \sum_{i=1}^{n} G_i,$$

where G_i are groups of type p^∞. Then there exist elements $b_{ij} \in G_j$, $j = 1, 2, \dots;\ i = 1, 2, \dots, n$, such that

$$pb_{i,j+1} = b_{ij}$$

and

$$pb_{i1} = 0.$$

It is obvious that any element $a \in I$ can be written uniquely in the form of a finite sum

$$\sum_{i,j} a_{ij} b_{ij},$$

where a_{ij} are integers, and moreover $|a_{ij}| \leq p/2$. To the element

$$a = \sum_{i,j} a_{ij} b_{ij}$$

we assign the number

$$\xi(a) = \sum_{i,j} \frac{|a_{ij}|}{p^j}.$$

For each finite subset $A \subset R$ and any natural number m, define the set

$$V_{A,m} = \{a \in I \mid \xi(a) \leq 1/2^m \text{ and } \xi(ab) \leq 1/2^m \text{ for any } b \in A\}.$$

It is proved that the collection $\{V_{A,m} \mid A \subset R,\ m = 1, 2, ...\}$ can be taken as a base of neighborhoods of zero, so as to turn R into a nondiscrete topological ring.

Recall that a subgroup H is called densely embedded in a group G if H has nonzero intersection with every nonzero subgroup of the group G.

Remark. It is obvious that every nonzero torsion-free group G contains an infinite densely embedded reduced subgroup H.

Lemma 4. *If a commutative ring R possesses a nonzero ideal I such that $I^2 = 0$ and I is a torsion-free group, then R admits a nondiscrete topologization in which I is an open ideal.*

Proof. Let H be a densely embedded reduced subgroup in the additive group of the ideal I. For any finite subset $M \subset R$ and any natural number n, define the set

$$V_{M,n} = \{a \in nH \mid aM \subset nH\}.$$

It is easy to verify that the collection $\{V_{M,n} \mid M \subset R,\ n = 1, 2, ...\}$ can be taken as a base of neighborhoods of zero, so as to turn R into a nondiscrete topological ring.

Lemma 5. *If I is an ideal of the ring R and B is the largest complete subgroup of the additive group of the ideal I, then B is an ideal in R.*

Lemma 6. *Let I be an ideal of the ring R, and let A be the set of all elements of finite order in I. If A has finite characteristic and R/A admits a nondiscrete topologization in which I/A is an open ideal, then R also admits a nondiscrete topologization in which I is an open ideal.*

Proof. Let n be the characteristic of the ring A, and let $\{\overline{V}_\alpha \mid \alpha \in \Omega\}$ be a base of neighborhoods of zero in the ring R/A. If φ is the canonical homomorphism of the ring R onto R/A, then put

$$U_\alpha = n\varphi^{-1}(\overline{V}_\alpha).$$

It is obvious that the collection of sets U_α, $\alpha \in \Omega$, can be taken as a base of neighborhoods of zero, so as to turn R into a nondiscrete topological ring, with I an open ideal.

Theorem 2. *Every commutative ring R containing an infinite nil ideal I admits a nondiscrete topologization in which I is an open ideal.*

Proof. By Theorem 1, R has an infinite ideal I_0 such that $I_0^2 = 0$. If A is the set of all elements of I_0 having finite order, then A is an ideal in R.

Consider two cases:

I. A is a finite ideal. Then $nA = 0$ for some number n. Since $\overline{R} = R/A$ contains an infinite ideal $\overline{I}_0 = I_0/A$ such that $\overline{I}_0^2 = 0$ and $\overline{I}_0$ contains no elements of finite order, it follows from Lemma 4 that $\overline{R}$ admits a nondiscrete topologization in which $\overline{I}_0$ is an open ideal. Then, by Lemma 6, R admits a nondiscrete topologization in which I_0 is an open ideal.

II. A is an infinite ideal. Let $\sum_i C_i$ be a decomposition of A into a direct sum of ideals C_i, the additive group of each of which is primary for some prime number p_i. If the sum $\sum_i C_i$ contains infinitely many summands, then the collection of ideals

$$V_n = \sum_{i=n}^{\infty} C_i$$

defines some nondiscrete topology in R. Suppose

$$A = \sum_{i=1}^{r} C_i,$$

then some C_{i_0} is infinite. If C_{i_0} contains infinitely many elements of order p_{i_0}, then, by Lemma 2, R admits a nondiscrete topologization in which I is open-

ideal. If, however, C_{i_0} contains only a finite number of elements of order p_{i_0}, then from Lemmas 1 and 5 it follows that R has such a nonzero ideal B that $B \subseteq I_0$ and the additive group of the ideal B is a finite direct sum of groups of type p^∞. By Lemma 3, R admits a nondiscrete topology in which I is an open ideal. This proves the theorem.

Lemma 7. *If a commutative ring R has an infinite ideal without nilpotent elements, then R admits a nondiscrete topology.*

Proof. If R has such an infinite ideal I that I contains no minimal ideals of the ring R, then there exists a decreasing transfinite sequence of nonzero ideals $I_1 \supset I_2 \supset \cdots$ in R such that $\bigcap_\alpha I_\alpha = 0$. Obviously, the collection of ideals I_α defines a nondiscrete topology in R.

Suppose now that every ideal of the ring R contains a minimal ideal of the ring R, and let I_1 be some infinite ideal without nilpotent elements in R. If I_1 satisfies the minimality condition for ideals, then I_1 is a finite direct sum of fields A_i. Since I_1 is an infinite ideal, some A_{i_0} is an infinite field. According to (6), a nondiscrete topology can be defined in A_{i_0}. From the fact that $R = A_{i_0} \oplus B$, it follows that a nondiscrete topology can also be defined in R. If, however, I_1 does not satisfy the minimality condition, then there exists an infinite number of fields $A_1, A_2, \ldots$ such that $A_i \subset I_1$ and the A_i are ideals in R. Then the collection of ideals

$$V_n = \sum_{i=n}^{\infty} A_i$$

defines a certain nondiscrete topology in R.

Lemma 8. *If an infinite commutative ring R of finite characteristic contains only a finite number of nilpotent elements and every nonzero ideal contains a nonzero nilpotent element, then R admits a nondiscrete topology.*

Proof. Without loss of generality, we may assume that the characteristic of the ring R is a power of a prime number p. For any subset $S \subseteq R$, by S^* we shall denote the annihilator of the set S in R.

If I is the set of all nilpotent elements of the ring R, then I is a finite ideal and, consequently, $I^{p^k} = 0$ for some number k. Consider the set $M = \{a \in R \mid \{a\}^* \cap C \nsubseteq I$ for any nonnilpotent ideal C of the ring R, contained in $I^*\}$. For any finite subset $Q \subset M$, define the set

$$V_Q = \left\{ \sum d_i^{p^k} \;\middle|\; d_i \in Q^*,\ pd_i = 0 \right\}.$$

It can be verified that the collection of sets V_Q, $Q \subset M$, defines a certain nondiscrete topology in R.

Lemma 9. *Every infinite commutative ring R of finite characteristic, containing only a finite number of nilpotent elements, admits a nondiscrete topology.*

Proof. Let $\mathfrak{N}$ be the set of all ideals in R that do not contain nilpotent elements, and let A be some maximal ideal in $\mathfrak{N}$. If A is an infinite ideal, then, by Lemma 7, R admits a nondiscrete topology. If, however, A is a finite ideal, then A has an identity element. Then $R = A \oplus B$. Since A is a maximal ideal in $\mathfrak{N}$, every ideal of the ring B contains a nonzero nilpotent element. By Lemma 8, B admits a nondiscrete topology. Then a nondiscrete topology can also be defined in R.

Theorem 3. *Every infinite commutative ring R admits a nondiscrete topology.*

Proof. If R contains an infinite number of nilpotent elements, then, by Theorem 2, R admits a nondiscrete topology.

Let now R contain only a finite number of nilpotent elements, and let A be the set of all elements of finite order of the ring R. Consider two cases:

I. A has finite characteristic. If $A = R$, then, by Lemma 9, R admits a nondiscrete topologization. If $A \neq R$, then from Theorem 2 and Lemmas 7 and 8 it follows that R admits a nondiscrete topologization.

II. A does not have finite characteristic. Let $\sum_i G_i$ be the decomposition of the additive group of the ideal A into primary subgroups G_i with respect to the prime numbers p_i. Then the G_i are ideals in R. From the fact that R contains only a finite number of nilpotent elements it follows that each G_i has finite characteristic. Hence $\sum G_i$ contains an infinite number of summands. It is clear that the collection of ideals

$$V_n = \sum_{i=n}^{\infty} G_i$$

defines a certain nondiscrete topology in R. This completes the proof of the theorem.

Institute of Mathematics
with Computing Center
Academy of Sciences of the MSSR
Kishinev

Received
6 III 1969

REFERENCES

1. V. I. Arnautov, DAN, 191, No. 4 (1970).
2. L. Fuchs, *Abelian Groups*, Budapest, 1958.
3. M. Hochster, Proc. Am. Math. Soc., 21, No. 2, 357 (1969).
4. J. O. Kiltinen, Trans. Am. Math. Soc., 134, No. 1, 149 (1968).
5. A. G. Kurosh, *Group Theory*, "Nauka," 1968.
6. A. F. Mutylin, Matem. zametki, 5, No. 2, 161 (1969).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.