THE BEHAVIOR OF EQUI-MORPHISMS AT INFINITY

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Abstract

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THE BEHAVIOR OF EQUI-MORPHISMS AT INFINITY

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In ([1]) it was established that every equi-morphism $f : \bar{H}^n \to H^{n'}$ of hyperbolic spaces preserves the tangency of curves with the absolute, as well as the tangency of curves lying on the absolute; however, the order of tangency with the absolute and the order of tangency of curves on the absolute, generally speaking, are not preserved. Here the properties of the equi-morphism f on the absolute will be clarified; namely, it turns out that although the order of tangency is not preserved, each of the three types of tangency—zero, finite, and infinite (see below)—remains invariant.

1. It is said that the tangency at a point a of two curves K and K_1 in a metric space R has order α, $0 \leq \alpha \leq \infty$, if for every $\alpha' < \alpha$ one can find a common parametrization of the curves (a one-to-one and continuous correspondence between the parameters) under which

$$xy/(ax)^{1+\alpha'} \to 0$$

($x \in K$, $y \in K_1$, $x, y \to a$), but for any $\alpha' > \alpha$ no such parametrization exists*. In particular, for $\alpha = 0$, $0 < \alpha < \infty$, $\alpha = \infty$, we obtain three types of tangency: zero, finite, and infinite.

Let us now consider a curve K_1 lying on the absolute S^{n-1} of the space H^n, and a curve K, touching it at a point $u \in S^{n-1}$, also lying on S^{n-1} or in H^n. Here the order of tangency of the curves K and K_1 is understood as the order of tangency of their images in the spherical Poincaré model (model P). If K_1 is the central projection of K onto the absolute, then the order of tangency of K with K_1 is called the order of tangency of K with the absolute.

Take a line leading to the point $u \in S^{n-1}$ as the axis oz of a "cylindrical" coordinate system in H^n: for any point $p \in H^n$ denote by d its distance from oz, and by z the projection of the vector $\overrightarrow{op}$ onto oz. Then the three introduced types of tangency of the curve K with the absolute can be distinguished, without resorting to a model, as follows.

Zero order of tangency: on K there exists a sequence of points $p_n \to u$ for which $d_n/z_n \to 0$ ($d_n \to \infty$, $p_n \to u$, $p_n \in K$); **finite order**: there exist two constants $c, C > 0$ such that $d/z > c$ for every point $p \in K$, but there exists a sequence $p_n \to u$ of points on K for which $d_n/z_n < C$; **infinite order**: $d/z \to \infty$ as $p \to u$, $p \in K$.

Consider, for example, a curve of zero order of tangency with the absolute. For it

$$(1-r)/\theta \to 0, \quad \theta \to 0,$$

but

$$(1-r)/\theta^{1+\alpha} \nrightarrow 0$$

for any $\alpha > 0$, where $r = \operatorname{th} \rho/2$, $\rho = op$, and θ is the angle between op and oz. The second relation means that

$$1/\operatorname{sh}\rho \cdot \theta^{1+\alpha} \nrightarrow 0$$

and, consequently,

$$\operatorname{sh}\alpha z/\operatorname{sh} d \nrightarrow 0$$

as $z \to \infty$, or

$$d - \alpha z \nrightarrow \infty$$

as $z \to \infty$. Hence it follows that for a sequence $a_n \to 0$ there is on the curve a sequence of points $p_n = (d_n, z_n)$ and a sequence $c_n > 0$ such that

$$z_n \to \infty, \quad c_n/z_n \to 0, \quad d_n - a_n z_n < c_n,$$

i.e.

$$d_n/z_n < a_n + c_n/z_n;$$

therefore $d_n/z_n \to 0$. For finite (infinite) order of tangency

$$\lim 1/\operatorname{sh}\rho \cdot \theta^{1+\alpha_0} = 0$$

for some $\alpha_0 > 0$ (for every $\alpha_0 > 0$), i.e. for all positive $\alpha \leq \alpha_0$,

$$d - \alpha z \to \infty,$$

whence, obviously, it follows that there is no sequence $p_n \in K$ for which $d_n/z_n \to 0$.

Remark. It is easy to see that if a curve K leading to the infinitely distant point u is subjected to a bounded shift, i.e. each of its points is moved a finite distance, then the order of its tangency with the absolute will not change.

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* For $\alpha = 0, \infty$ this formulation must be modified in a natural way.

The following theorems establish the invariance of each of the three indicated types of tangency.

2. Theorem 1. *The zero order of tangency of a curve K with the absolute is invariant under equimorphisms of the space H^n.*

Proof. As has just been established, for some sequence p_n of points of the curve, $d_n/z_n \to 0$ as $p_n \to u$. We shall show, first, that d'_n/d_n remains bounded on both sides under any equimorphism f. Denote by q_n the projection of the point p_n onto oz, the images of the points p_n and q_n under the equimorphism f by p'_n and q'_n, and the least distance of the point p'_n from the image of the axis oz by D'_n. Then $D'_n \leq p'_n q'_n \leq L \cdot p_{nq} n = L \cdot d_n$ (for sufficiently large distances the equimorphism satisfies a two-sided Lipschitz condition; the constants entering here will be denoted by l and L). Further, $d'_n \leq D'_n + c_0$, where c_0 is the greatest deviation of the image of oz from oz (see (2)); and we assume that the points u and o remain fixed. Thus

$$d'_n \leq L \cdot d_n + c_0 < (L+1)d_n = \tilde{L} \cdot d_n.$$

Considering the inverse mapping f^{-1}, in the same way we verify that $d'_n > \tilde{l} \cdot d_n$. Hence

$$\tilde{l} < d'_n/d_n < \tilde{L},$$

where $\tilde{l}, \tilde{L} > 0$ are certain constants. We now finish the proof of the invariance of the zero order of tangency. Similarly to the preceding, we have

$$l < \rho'_n/\rho_n < L,$$

where $\rho_n = op_n,\ \rho'_n = op'_n$. Hence

$$l/2 < (z'_n + d'_n)/(z_n + d_n) < 2L,$$

whence it follows that

$$\frac{l}{2} < \frac{z'_n/z_n + d'_n/z_n}{1 + d_n/z_n} < 2L,$$

and since $d_n/z_n \to 0$ and $d'_n/z_n \to 0$ (the latter by virtue of the boundedness of the ratio d'_n/d_n) as $n \to \infty$, it follows that

$$l/3 < z'_n/z_n < 3L.$$

Therefore

$$\frac{d'_n}{z'_n} = \frac{d'_n}{d_n}\frac{d_n}{z_n}\frac{z_n}{z'_n} \to 0,$$

as was required.

Before formulating Theorem 2, let us prove one auxiliary proposition, which will also be needed later.

Lemma. *In order that two curves $K_1, K_2 \subset S^{n-1}$ have order of tangency at the point u: a) zero, b) infinite, it is necessary and sufficient that on each projecting cone oK_1 and oK_2 there be found two curves $\mathcal{K}_1$ and $\mathcal{K}_2$, deviating boundedly* *from one another, such that: a) $\mathcal{K}_1$ and $\mathcal{K}_2$ touch the absolute; any two such curves have order of tangency with the absolute not exceeding zero; b) $\mathcal{K}_1$ and $\mathcal{K}_2$ have infinite order of tangency with the absolute.*

Proof. Case a). Necessity. Let the order of tangency of K_1 with K_2 be equal to zero. Let $u_1 \in K_1,\ u_2 \in K_2$. If by β we denote the angle $u_1 o u_2$, and by ψ_1 and ψ_2 the angles $u_1 o u$ and $u_2 o u$, respectively, then

$$\lim \beta/\psi_i = 0; \qquad \beta/\psi_i^{1+\alpha} \nrightarrow 0, \quad i = 1, 2;\ \alpha > 0.$$

On the projecting cone oK_1 construct a curve $\mathcal{K}_1$ with equation $1 - r_1 = \beta$. Then

$$(1 - r_1)/\psi_1 = \beta/\psi_1 \to 0$$

and

$$(1 - r_1)/\psi_1^{1+\alpha} \nrightarrow 0,$$

which means a zero order of tangency of K_1 with the absolute. In exactly the same way, on oK_2 construct a curve $\mathcal{K}_2$ with equation $1 - r_2 = \beta$ (i.e., put $r_1 = r_2 = r = 1 - \beta$, where $r = \operatorname{th} \rho/2$). Denote the distance between corresponding points on $\mathcal{K}_1$ and $\mathcal{K}_2$ by d. Then

$$\operatorname{sh} \frac{d}{2} = \operatorname{sh} \rho \, \sin \frac{\beta}{2} = \frac{2r}{1 + r} \frac{\sin \beta/2}{1 - r} \to \frac{1}{2}$$

as $u_{1,2} \to u$.

Now suppose that there are $\overline{\mathcal{K}}_1 \subset oK_1$ and $\overline{\mathcal{K}}_2 \subset oK_2$, boundedly deviating from each other, and that at least one of them has order of tangency with the absolute greater than zero. Then, by the remark to item 1, it follows that the second curve will have the same order; denote it by α_0. Let the boundedness of the distance between $\overline{\mathcal{K}}_1$ and $\overline{\mathcal{K}}_2$ be attained under the correspondence $x_1 \leftrightarrow x_2$ ($x_1 \in \overline{\mathcal{K}}_1,\ x_2 \in \overline{\mathcal{K}}_2$). From the triangle $o x_1 x_2$ we have

$$\beta^2 \leq c(1 - r_1)(1 - r_2), \qquad c = \text{const}.$$

Consider the ratio $\beta/\psi_1^{1+\alpha}$, where $0 < \alpha < \alpha_0$. Then

$$\frac{\beta^2}{\psi_1^{(1+\alpha)2}} \leq \frac{c(1 - r_1)(1 - r_2)}{\psi_1^{2(1+\alpha)}} = c \, \frac{1 - r_1}{\psi_1^{1+\alpha}} \frac{1 - r_2}{\psi_1^{1+\alpha}}.$$

Since ψ_1 and ψ_2 are equivalent infinitesimals (see (1)), both of the last factors tend to zero. Therefore $\beta/\psi_1^{1+\alpha} \to 0$, and hence the order of tangency of K_1 with K_2 is not less than α_0.

Sufficiency. Instead of the required assertion, we shall give the formulation of an assertion equivalent to it. If K_1 and K_2 have at the point $u \in S^{n-1}$ contact of finite (or more than finite) order, then on the projecting cones oK_1 and oK_2 one can find curves $\mathcal{K}_1$ and $\mathcal{K}_2$ having finite order of contact with the absolute and deviating from one another in a bounded manner. The proof of this fact is entirely analogous to the preceding proof.

Case b). Necessity. Just as in the preceding case, we construct curves $\mathcal{K}_i$, $1-r_i = \beta$, $i = 1,2$, which, obviously, have infinite order of contact with the absolute.

Sufficiency. Suppose that on oK_1 and oK_2 there are curves $\mathcal{K}_1$ and $\mathcal{K}_2$ at a bounded distance from one another, the order of contact of each of them with the absolute being infinite. Taking, as above, $x_1 \leftrightarrow x_2$ as the correspondence between these curves and considering the triangle ox_1x_2, we obtain, as before, $\beta^2 \leq c(1-r_1)(1-r_2)$, where $(1-r_1)/\psi_1^N \to 0$ and $(1-r_2)/\psi_2^N \to 0$ for any N greater than or equal to one. In this case $\beta^2/\psi_1^{2N} \leq c(1-r_1)(1-r_2)/\psi_1^{2N} \to 0$, as was required.

Theorem 2. *The zero order of contact of curves K_1 and K_2 lying on the absolute is invariant with respect to equimorphisms of the space H^n.*

Proof. Denote the images of the curves K_1 and K_2 under the equimorphism f by K_1' and K_2'. Suppose that the order of contact of K_1' with K_2' is finite (or infinite). Then on the projecting cones $o'K_1'$ and $o'K_2'$ there will be found curves $\mathcal{K}_1'$ and $\mathcal{K}_2'$, deviating from one another in a bounded manner and having finite order of contact with the absolute. In this case the equimorphism f^{-1} takes $\mathcal{K}_1'$ and $\mathcal{K}_2'$ into curves $\widetilde{\mathcal{K}}_1$ and $\widetilde{\mathcal{K}}_2$ with finite (nonzero) order of contact with the absolute. Projecting orthogonally the curves $\widetilde{\mathcal{K}}_1$ and $\widetilde{\mathcal{K}}_2$ onto the cones oK_1 and oK_2, respectively, we obtain curves $\mathcal{K}_1$ and $\mathcal{K}_2$, lying at a bounded distance from $\widetilde{\mathcal{K}}_1$ and $\widetilde{\mathcal{K}}_2$ and, consequently, having at least finite order of contact with the absolute. If the distance between $\mathcal{K}_1$ and $\mathcal{K}_2$ is bounded, then the curves K_1 and K_2 could not have contact of zero order; if, however, it is unbounded, then $\mathcal{K}_1'$ and $\mathcal{K}_2'$ also could not lie at a finite distance from one another. The theorem is proved.

3. Theorem 3. *The infinite order of contact with the absolute of a curve K leading to an infinitely remote point u is an invariant of equimorphisms of the space H^n.*

The **proof** rests on somewhat different considerations than the proof of Theorem 1.

Theorem 4. *The infinite order of contact of curves K_1 and K_2 lying on the absolute is invariant with respect to equimorphisms of H^n.*

Proof. Denote the images of the curves K_1 and K_2 by K_1' and K_2'. By virtue of the lemma of paragraph 1, on the projecting cones oK_1 and oK_2 there exist curves $\mathcal{K}_1$ and $\mathcal{K}_2$, deviating from one another in a bounded manner and having infinite order of contact with the absolute. This means that the projections of their images $\mathcal{K}_1'$ and $\mathcal{K}_2'$ onto the projecting cones $o'K_1'$ and $o'K_2'$ deviate from one another in a bounded manner and also touch the absolute with infinite order of contact. From the already mentioned lemma it follows that the order of contact of the curves K_1' and K_2' is infinite, as was required.

From the preceding theorems follows the validity of the following theorem.

Theorem 5. *The finite order of contact of a curve with the absolute, as well as the finite order of contact of two curves lying on the absolute, remains finite for their images under equimorphisms of the space, i.e., finiteness of the order of contact is invariant with respect to equimorphisms of the space* H^n*.*

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CITED LITERATURE

1. V. A. Efremovich, V. I. Pupko, DAN, 160, No. 1 (1965).

2. V. A. Efremovich, E. S. Tikhomirova, DAN, 152, No. 5 (1963).

Note: Figure translations are in progress. See original paper for figures.