ESTIMATION OF THE SPECTRAL DENSITY OF A RANDOM FIELD FROM ITS LINEAR SECTIONS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 519.24

MATHEMATICS

K. V. KONIAEV

ESTIMATION OF THE SPECTRAL DENSITY OF A RANDOM FIELD FROM ITS LINEAR SECTIONS

(Presented by Academician A. N. Kolmogorov on 27 III 1970)

For the spectral analysis of fields defined in spaces of two or more dimensions, when using known estimates of the spectrum it is necessary to have realizations with the full number of dimensions. If this requirement is not satisfied—for example, if a realization of a plane (two-dimensional) field is measured along two intersecting linear sections—then the corresponding spectral window (smoothing function) has the form of two intersecting ridges with a peak in the region of their intersection. It is desirable to work with a spectral window in the form of a single narrow peak, which requires measuring the plane field within some area. As a rule, random fields are measured not in the space of all dimensions, but within linear or plane sections. For example, temperature pulsations in the ocean are measured by point sensors along the course of a vessel or by several fixed sensors installed along one or two lines. Therefore the known estimates of the spectrum of multidimensional fields prove to be of little practical use ([1]).

It is possible to construct an estimate of the spectrum that permits analysis with an acceptable spectral window from data on sections of the field. Let a realization of a plane homogeneous ergodic random field with zero mean have the form of two linear sections of finite extent along the axes x and y:

$$\xi_a(x, y)[\delta(x) + \delta(y)],$$

where $\xi_a(x, y)$ is a realization of the field within a bounded area; $\delta(x)$, $\delta(y)$ are delta functions. The usual estimate of the correlation function in this case consists of four terms:

$$
\begin{aligned}
\hat{B}(\Delta x, \Delta y) = \frac{1}{a} \int \xi_a(x,y)[\delta(x) + \delta(y)]\xi_a(x - \Delta x, y - \Delta y)[\delta(x - \Delta x) +\\
+ \delta(y - \Delta y)]\, dx\, dy\\
= \frac{1}{a}\Big[\delta(\Delta x) \int \xi_a(0,y)\xi_a(-\Delta x, y - \Delta y)\, dy +\\
+\delta(\Delta y) \int \xi_a(x,0)\xi_a(x - \Delta x, -\Delta y)\, dx\\
+\xi_a(0,\Delta y)\xi_a(-\Delta x, 0) + \xi_a(\Delta x, 0)\xi_a(0, -\Delta y)\Big],
\end{aligned}
\tag{1}
$$

where $1/a$ is a normalizing factor. The first two terms estimate two sections of the correlation function along the axes Δx and Δy. The remaining two terms make it possible to estimate the correlation function within a rectangular area whose linear dimensions are equal to the linear dimensions of the two original sections. In the usual estimate of the spectrum (the Fourier transform of the estimate of the correlation function), all the terms listed are used. One may retain in (1) only the last two terms; then the estimate of the spectrum obtained by Fourier transforming these terms, pri-

takes the form *:

$$
\hat{g}(k,l) = \frac{2}{a}\,\mathrm{Re}\,[c_x(k)c_y(-l)], \tag{2}
$$

where Re denotes the real part; $c_x(k) = \int \xi_a(x,0)\exp(-i2\pi kx)\, dx$, $c_y(-l) = \int \xi_a(0,y)\exp(i2\pi ly)\, dy$ are the amplitude spectra of two linear sections of the field; k, l are the components of the spatial frequency (wave vector).

The mathematical expectation of the estimate (2), $E\hat{g}(k,l)$, is equal to the smoothed spectral density $g(k,l)$ of the field

$$
E\hat{g}(k,l) = \frac{2}{a}\,\mathrm{Re} \int g(m,n)h(k-m, n-l)\, dm\, dn. \tag{3}
$$

The smoothing function, or spectral window, $h(k,-l)$, is equal to the product of the random smoothing functions $h_x(k)h_y(-l)$, where $h_x(k)$ and $h_y(-l)$ are the Fourier transforms of functions equal to unity on the corresponding linear sections of the field and to zero outside these sections (as usual, it is possible to introduce weight functions for realizations). The spectral window $h(k,-l)$ has an acceptable single-peak form if the sections of the field are approximately symmetric with respect to the point of intersection. The principal side lobes of the spectral window lie on the axes k and l. As the length of the linear sections increases, the spectral window $h(k,-l)$ tends to the product of two delta functions $\delta(k)\delta(l)$, and the mathematical expectation of the estimate (2) tends to the true value of the spectral density. Thus, the estimate (2) is asymptotically unbiased.

For a normal random field, the variance of the estimate (2), $D\hat{g}(k,l)$, can be expressed in terms of the spectral density $g(k,l)$ of the field, using the representation of the fourth moment of the field by pairwise products of second moments:

$$D\hat{g}(k,l) = \frac{2}{a^2}\{ \operatorname{Re}\left|\iint g(m,n)h_x(k-m)h_y(n-l)\,dm\,dn\right|^2 + \\ + \left|\iint g(m,-n)h_x(k-m)h_y(l-n)\,dm\,dn\right|^2 + \\ + \int g_x(m)|h_x(k-m)|^2\,dm \int g_y(n)|h_y(l-n)|^2\,dn + \\ + \operatorname{Re}\int g_x(m)h_x(k-m)h_x(k+m)\,dm \int g_y(n)h_y(n-l)h_y(-l-n)\,dn\}, \tag{4}$$

where

$$g_x(k) = \int B(\Delta x, 0)\exp(-i2\pi k\Delta x)\,d\Delta x, \qquad g_y(l) = \int B(0,\Delta y)\exp(-i2\pi l\Delta y)\,d\Delta y.$$

The first term corresponds to the usual dependence of the variance of the estimate at any point k, l on the smoothed spectral density at this point. In addition, the variance of the estimate (2) at the point k, l depends ** on the integral variance of the spectral components of the field within two mutually perpendicular bands passing through the point k, l, and on the spectral density at the point $k, -l$.

The relative magnitude of the variance of the estimate (2) is large, since the estimation of each value of the correlation function in (1) within the area is performed from a single pair of values of the random field (the estimate (2) is inconsistent). As usual, the variance of the estimate can be reduced by averaging the estimate over many realizations or over space–

* In radio astronomy, cross-shaped radio interferometers are used (Mills crosses), whose operation can be reduced to an algorithm of type (2). However, the Mills cross can perform only sequential analysis of the spectrum because of the presence of integration over the antenna aperture.

** Radio astronomers apparently do not pay attention to the unusual behavior of the variance of the estimate, achieving comparatively accurate estimation by means of extensive averaging of the estimate over time.

···frequencies. Averaging over frequencies must be carried out after computing the unsmoothed estimate (2).

The spectrum can be estimated from any two nonparallel linear sections. As the angle between the sections is decreased, the main peak of the spectral window in (3) broadens.

The estimate of the spectrum still has the form of a product of the spectra of the linear sections; however, if the sections are given discretely at a number of points, then the estimate is determined at the nodes of an oblique-angled grid, with the principal axes forming the same angle between them as the original sections, while the generators of the oblique-angled grid are perpendicular to these axes.

Fig. 1. Directionally unambiguous spatial spectrum $g(k,l)$, (mm of level $\cdot$ m)^2, of sea waves, estimated from data from two moving sensors. Points with numbers show the computed values of the spectral estimate. Horizontals with numbers are constructed by means of linear interpolation of the computed spectral values and correspond to lines of constant spectral density.

If two sections have no common point and are separated from each other, then the spectral window acquires an oscillatory character*.

The sections of the field need not necessarily be purely spatial or purely temporal, but may be made along inclined trajectories in space-time coordinates, which corresponds to measuring the field with point moving sensors.

It is easy to obtain an estimate of type (2) for a field specified in a space of n dimensions. The spectrum of an n-dimensional random field can be estimated from two sections of the field whose total dimensionality is not less than the dimensionality of the field. For example, for a three-dimensional field the spectral estimate can be computed from a realization measured on two planes, and then, up to a normalizing factor, the estimate has the form

$$\hat{g}(k,l,m) = \mathrm{Re}\left[c_{x,y}(k,l)c_{y,z}(-l,-m)\right].$$

---

* Windows of this type are used in optical and radio interferometers to measure the angular sizes of remote radiation sources.

or from a realization measured on a plane area and a segment of a straight line:

$$\hat{g}(k,l,m) = \mathrm{Re}[c_x(k)c_{y,z}(-l,-m)],$$

where $c_x(x)$, $c_{x,y}(k,l)$, $c_{y,z}(-l,-m)$ are the Fourier transforms, respectively, of one- and two-dimensional realizations of the three-dimensional field.

For a Mills cross (a four-dimensional field) we have:

$$\hat{g}(k, l, f_0) = \mathrm{Re}[c_{x,t}(k, f_0) c_{y,t}(-l, -f_0)].$$

The coordinate m is determined by the coordinates k, l, f_0 owing to the relation between the modulus of the spatial frequency and the temporal frequency.

Estimate (2) was used to estimate the spatial spectrum of sea waves, unambiguous with respect to direction, from data of two moving sensors. The wave motion is a random field defined on the plane x, y and in time t (a three-dimensional field), and the spectral densities of the wave motion are nonzero on the surface of two parabolic cones in the frequency space k, l, f:

$$(k^2 + l^2)^{1/2} = 2\pi f^2 / g, \tag{5}$$

where $g = 9.8$ m/sec^2.

The trajectories of the moving sensors form a cross in an inclined plane in the space x, y, t. Therefore relation (2) makes it possible to estimate the projection of the wave spectrum onto an inclined plane in the space k, l, f. Projections of the spectrum onto several inclined planes make it possible to determine the projection of the spectrum onto the plane k, l; moreover, in order to preserve unambiguity with respect to the direction of wave motion, the projection onto the plane k, l is made only from one of the two parabolic cones (5). The wave spectrum measured in this way is shown in Fig. 1. The spectrum clearly shows a rapid broadening of the angular distribution and a decrease in the spectral density as the modulus of the spatial frequency increases.

Acoustics Institute
Moscow

Received
24 II 1970

REFERENCES

1. U. Grenander, M. Rosenblatt, *Collected Translations. Mathematics*, **2**, 5, 105 (1958).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.