

ON THE COMPLEXITY OF REALIZING PROPOSITIONAL FORMULAS

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Abstract

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MATHEMATICS

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ON THE COMPLEXITY OF REALIZING PROPOSITIONAL FORMULAS

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In paper ⁽⁹⁾ results were obtained showing that, along with propositional formulas that have no upper bound for the complexity of realizability, there are also formulas for which the function $\lambda x C_1 \cdot 14^x + C_2^*$, for certain C_1 and C_2 , is an upper bound for the complexity of realizability. The question arises: is it true that every propositional formula either has no upper bound for the complexity of realizability, or the function $\lambda x C_1 \cdot 14^x + C_2$ is its upper bound for the complexity of realizability?

In the present paper results will be presented that concern only part of the question posed: a condition will be formulated sufficient for a propositional formula not to have an upper bound for the complexity of realizability; an affirmative answer will be given to the question posed in the case when the propositional formula is a formula in one variable.

We shall use the terminology and concepts introduced in papers ⁽¹⁻⁶⁾, and shall also assume that the reader is familiar with paper ⁽⁹⁾.

1. In ⁽⁹⁾ concepts concerning propositional formulas are introduced for formulas in one variable. We shall extend them to an arbitrary propositional formula.

Let k be a natural number, and let Q be a k -term system of closed logical-arithmetical formulas. By the symbol Q_i ($1 \leq i \leq k$) we shall denote the i -th term of the system Q . Let n be a natural number, and let G be a propositional formula in the variables p_1, \dots, p_k . **We shall say that a normal algorithm \mathfrak{A} over the alphabet A^* n -realizes the formula G** if, for every k -term system of closed logical-arithmetical formulas Q such that $|Q_i^o| \leq n$ ($1 \leq i \leq k$),

- a) the algorithm \mathfrak{A} is applicable to Q ;
- b) $\mathfrak{A}(Q)$ realizes the formula $G(Q_1, \dots, Q_k)$.

Let G be a propositional formula. We shall say that G has no upper bound for the complexity of realizability if, for any general recursive function f , there exists a natural number N such that for every natural number m such that

$m \geq N$, and every normal algorithm \mathfrak{A} in the alphabet A^+ that m -realizes G , one has $\mathfrak{A}' \geq f(m)$.

* Here and below we use λ -notation (see ⁽²⁾, p. 37) for functions.

** A propositional formula G is a formula in the variables p_1, \dots, p_k if p_i ($1 \leq i \leq k$) occurs in G and no propositional variable other than p_1, \dots, p_k occurs in G .

*** In the present paper, as in ⁽⁹⁾, A is the alphabet of logical-arithmetical formulas, and A^+ is the standard two-letter extension of the alphabet A .

We note that Theorem 5 of paper ⁽⁹⁾ remains true also when F and G are arbitrary propositional formulas, and not formulas in one variable, as is required in paper ⁽⁹⁾.

2. Let J_1 be the first Jaśkowski matrix (see ⁽⁶⁾, p. 7).

Theorem 1. *If a propositional formula G is refutable on J_1 , then G has no upper bound for realizability complexity.*

For the proof of this theorem one uses Theorem 4 of paper ⁽⁹⁾, as well as the strengthening, mentioned above, of Theorem 5 of paper ⁽⁹⁾, and a fragment of the proof of Theorem 1 of paper ⁽⁸⁾ (if G is refutable on J_1 , then $G \vdash \neg\neg p \supset p$ in the intuitionistic propositional calculus with the substitution rule).

3. Let G be a propositional formula, and let f be a general recursive function. We shall say that f is an upper bound for the realizability complexity of the formula G , if for every natural number n there exists a normal quasi-algorithm \mathfrak{A} in the alphabet A^+ such that \mathfrak{A} n -realizes G and $\mathfrak{A}_3 \leq f(n)$.

Theorem 2. *Let G_1, \dots, G_k and F be propositional formulas, and let f_1, \dots, f_k be general recursive functions such that f_i is an upper bound for the realizability complexity of G_i ($1 \leq i \leq k$), and suppose that the formula $(G_1 \& \dots \& G_k) \supset F$ is derivable in the intuitionistic propositional calculus. Then there exist natural numbers C_0, C_1, \dots, C_k such that the function*

$$\lambda x \sum_{i=1}^k C_i \cdot f_i(x) + C_0$$

is an upper bound for the realizability complexity of F .

We shall say that a propositional formula G has a trivial upper bound for realizability complexity if, for some natural numbers C_1 and C_2 , the function $\lambda x C_1 \cdot 14^x + C_2$ is an upper bound for the realizability complexity of G .

From Theorem 1 of paper ⁽⁷⁾, Theorem 9 of paper ⁽⁹⁾, and Theorems 1 and 2 of the present paper, the following can be obtained:

Theorem 3. *Every propositional formula G in one variable either has no upper bound for realizability complexity, or has a trivial upper bound for realizability complexity.*

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REFERENCES

- ¹ A. A. Markov, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **42** (1954).
- ² S. K. Kleene, *Introduction to Metamathematics*, Moscow, 1957.
- ³ N. A. Shanin, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **43** (1955).
- ⁴ A. A. Markov, Izv. AN SSSR, ser. matem., **31**, No. 1, 161 (1967).
- ⁵ S. C. Kleene, J. Symbolic Logic, **10**, 109 (1945).
- ⁶ G. F. Rose, Trans. Am. Math. Soc., **75**, 1 (1953).
- ⁷ I. Nishimura, J. Symbolic Logic, **25**, 327 (1960).
- ⁸ V. A. Yankov, Izv. AN SSSR, ser. matem., **32**, No. 1, 208.
- ⁹ V. I. Khomich, DAN, **191**, No. 5, 1004 (1970).

Note: Figure translations are in progress. See original paper for figures.

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