CLASSES OF CONJUGATE SUBGROUPS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 519.46

MATHEMATICS

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CLASSES OF CONJUGATE SUBGROUPS IN TOPOLOGICAL GROUPS

(Presented by Academician P. S. Novikov on 26 V 1969)

Let G be an arbitrary topological group, and H its closed subgroup. Denote by S_H the class of subgroups of the group G conjugate to H. On the set S_H, whose elements are the subgroups $g^{-1}Hg$, one can introduce a topology by declaring a neighborhood of the element $g^{-1}Hg$ to be the collection of subgroups of the form $u^{-1}Hu$, where u runs through some neighborhood U of the element g. We obtain a topological space which, as is easy to show, is homeomorphic to the quotient space G/N of the group G by the normalizer $N = N(H)$ of the subgroup H (the normalizer of a closed subgroup is closed—see ([3])), and the homeomorphism is given as follows:

$$x^{-1}Hx \longleftrightarrow Nx.$$

In the author' s paper ([2]), groups were studied for which all classes of conjugate subgroups are bicompact. In the case of discrete groups this condition becomes the condition that the classes of conjugate subgroups be finite. Such groups were considered by Neumann ([1]).

In the present paper we shall consider groups in which only the classes of conjugate **bicompact** subgroups are bicompact. Some results of ([2]) admit strengthening, as is shown by the

Theorem. 1) *If G is a locally bicompact group in which all classes of conjugate bicompact subgroups with one topological generator are bicompact, then the set P of all bicompact elements of the group G forms a closed locally normal subgroup (the periodic part of G), the quotient group by which is pure, i.e. contains no bicompact elements distinct from the identity.*

2) *If all classes of conjugate bicompact subgroups of a locally bicompact group G are bicompact, then G is projectively Lie, i.e. in every neighborhood of its identity there is a bicompact normal divisor such that the quotient group by it is a Lie group.*

3) *If in a locally bicompact group G the classes of conjugate subgroups with one topological generator are bicompact, then G is an $\overline{FS}$-group, i.e. the closures of all its classes of conjugate elements are bicompact.*

Recall that a bicompact element is an element contained in a bicompact subgroup, while a locally normal group is a group each element of which lies in a bicompact normal divisor.

For the proof of the theorem we shall need the

Lemma. *Let H be a bicompact subgroup of a locally bicompact group G, and let the space S_H be bicompact. Then H is contained in a bicompact normal divisor of the group G.*

Proof. Let H be bicompact and let N be the normalizer of H, $\tilde{G} = G/N$. By virtue of the homeomorphism of S_H and $\tilde{G}$, the quotient space $\tilde{G}$ is bicompact. Take in G a neighborhood U of the identity with bicompact closure $\overline{U}$. By $\overline{Ug}$ denote the set Nug of right cosets, where u runs through the neighborhood U. From the covering of the bicompact set $\tilde{G}$

by the domains $\widetilde{U}g$, choose a finite subcovering $\widetilde{U}g_1, Ug_2, \dots, \widetilde{U}g_k$. Then the group G will be the set-theoretic sum of its subsets $NUg_1, \dots, NUg_k$. Let g be an arbitrary element of G. It has the form xug_i, where $x \in N,\ u \in U,\ 1 \le i \le k$. Then

$$g^{-1}Hg = g_i^{-1}u^{-1}x^{-1}Hxug_i = g_i^{-1}u^{-1}Hug_i.$$

It follows that the set $B = \bigcup g^{-1}Hg$ is contained in the bicompact set

$$C = \bigcup_{i=1}^{k} g_i^{-1}\overline{U}^{-1}H\overline{U}g_i$$

and therefore has bicompact closure. Further, B is invariant and consists of bicompact elements. Finally, in [4] (see Theorem 3) it was shown that every invariant set B of a locally bicompact group G, consisting of bicompact elements and having bicompact closure, generates, in the topological sense, the bicompact normal divisor $\overline{\{B\}}$ of the group G (in Theorem 3 of [4] it was required that the set B itself be bicompact, but this requirement can be weakened, and no changes in the proof are needed). The inclusion $H \subset \overline{\{B\}}$ proves the lemma.

We pass to the proof of the first assertion of the theorem. Let a_1 and a_2 be two bicompact elements of the group G, $A_1 = \{a_1\}$, $A_2 = \{a_2\}$. By the lemma, A_1 and A_2 are contained in bicompact normal divisors N_1 and, respectively, N_2 of the group G. Then the element a_1a_2 is contained in the bicompact normal divisor N_1N_2, whence it follows that the set P is a subgroup in the algebraic sense. The closedness of P and the purity of G/P follow from (5).

Let us prove the second assertion of the theorem. It is known (see [6], Theorem 9) that a locally bicompact group G contains an open projective-left subgroup H. Choose, in an arbitrary neighborhood $U \subset H$ of the identity, a bicompact normal divisor B of the group H such that the quotient group H/B is a Lie group. The normalizer N of the subgroup B contains H, and therefore is open in G and, defining a bicompact quotient space, has finite index in G. The intersection N' of all subgroups conjugate to N will be an open normal divisor of the group G of finite index. The subgroup $B \cap N'$ is invariant in N'.

Consider the quotient group $H \cap N'/B \cap N'$. It is locally bicompact and admits a continuous one-to-one homomorphism into the Lie group H/B:

$$H \cap N'/B \cap N' = H \cap N'/B \cap (H \cap N') \cong (H \cap N')B/B \subset H/B.$$

By Cartan' s theorem (see [7], pp. 190-198), $H \cap N'/B \cap N'$ is a Lie group, and since the subgroup $H \cap N'$ is open in N', it follows that $N'/B \cap N'$ is also a Lie group. We thus have an invariant open subgroup N' in G and a bicompact normal divisor $B' = B \cap N'$ of the group N', lying in the neighborhood U and defining the Lie quotient group N'/B'. The normalizer of the subgroup B' contains N' and, consequently, has finite index. This means that B' has a finite number of subgroups conjugate to it,

$$g_1^{-1}B'g_1, \dots,\ g_n^{-1}B'g_n.$$

From the invariance of N' in G it follows that the transformation of N' by elements of G is its topological automorphism. Hence all the subgroups $g_i^{-1}B'g_i,\ i = 1, \dots, n$, are invariant in N' and define Lie quotient groups. The intersection B'' of these subgroups will be a normal divisor in G. The quotient group N'/B'' is a Lie group by Lemma 3 of [6]. Since G/N' is a discrete group, G/B'' is a Lie group.

We shall prove the last assertion of the theorem. If g is a bicompact element of the group G, i.e., the subgroup $\{g\}$ is bicompact, then, by the lemma, $\overline{\{g\}}$ is contained in a bicompact normal divisor of the group G, whence it follows that g is an FC-element. If the element g is not bicompact, then it is pure, i.e., it generates a discrete (and therefore closed) infinite cyclic subgroup $\{g\}$. By hypothesis, the factor space G/N, where N is the normalizer of $\{g\}$, is bicompact. Transformation by means of elements of N induces in $\{g\}$ some automorphism, and under an automorphism the property of an element of being a generator is preserved. Since in $\{g\}$ there exist only two generating elements, g and g^{-1}, g has in N at most two conjugate elements, whence it follows that $N/Z_N(g)$ is a finite factor space (by $Z_N(g)$ we denote the centralizer of $\{g\}$ in N). Since, further, $Z_N(g) = Z_G(g)$, the factor space $G/Z_G(g)$ is bicompact. In that case the class of elements conjugate to g, homeomorphic to this factor space, is also bicompact. Thus the pure elements have turned out to be even FC-elements, i.e., to determine bicompact classes of conjugate elements. Analogous arguments show that all elements of finite order are also FC-elements. The theorem is completely proved.

From the first assertion of the theorem we obtain

Corollary. *A periodic* locally bicompact group in which the bicompact classes of conjugate subgroups topologically generated by one element are bicompact is locally normal.*

In the case of discrete groups this assertion admits a converse: in a locally normal discrete group all classes of conjugate finite subgroups are finite. In the general case such a converse is impossible, as is shown by the example, cited in ([8]), of a topological wreath product of a cyclic group of order 2 and a group of type 2^∞.

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Received
11 V 1969

CITED LITERATURE

[1] B. H. Neumann, Math. Zs., **63**, No. 1, 76 (1955).
[2] V. I. Ushakov, Matem. sborn., **63**, 2, 277 (1964).
[3] A. G. Kurosh, Izv. AN SSSR, ser. matem., **9**, 65 (1945).
[4] V. I. Ushakov, Sibirsk. matem. zhurn., **4**, No. 5, 1162 (1963).
[5] V. P. Platonov, ibid., **7**, No. 4, 854 (1966).
[6] V. M. Glushkov, UMN, **12**, issue 2 (74), 3 (1957).
[7] C. Chevalley, *Theory of Lie Groups*, 1, IL, 1958.
[8] V. I. Ushakov, Sibirsk. matem. zhurn., **4**, No. 3, 689 (1963).

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* A topological group is called periodic if all its elements are bicompact.

Note: Figure translations are in progress. See original paper for figures.