Reports of the Academy of Sciences of the USSR

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1970. Volume 191, No. 4

UDC 517.945.43

MATHEMATICS

B. M. BUDAK, M. Z. MOSKAL

ON A CLASSICAL SOLUTION OF A MULTIDIMENSIONAL MULTIPHASE PROBLEM OF STEFAN TYPE IN A DOMAIN WITH PIECEWISE SMOOTH BOUNDARY

(Presented by Academician A. N. Tikhonov on 16 VII 1969)

This paper considers questions of existence, uniqueness, stability with respect to perturbations of the initial data, and smoothness of the solution of a multidimensional, multiphase problem of Stefan type for a general linear parabolic equation of the second order, with nonclosed and closed fronts in noncylindrical domains with piecewise smooth boundary. The exposition is given for the case when the number of spatial independent variables is $N = 2$, but the method and results are easily extended to the case $N > 2$. For definiteness, the first boundary-value problem is discussed; the second boundary-value problem is considered analogously.

Statement of the problem. As the basic domain of variation of the independent variables x_1, x_2, t, we take the domain D_T, bounded by the surfaces: $\sigma_T^{n+i} \equiv \{x_1 = x_1(s,t), x_2 = x_2(s,t)\}$, s is a parameter, t is time, $s \in [0, s_{n+i}]$, $i = 1, 2, 3$, $t \in [0,T]$, and by the planes $t = 0$, $t = T$, $x_1 = P = \text{const}$, arranged as follows:

$$\sigma_T^{n+2} \cap \sigma_T^{n+3} = \emptyset \quad (\emptyset \text{ is the empty set}), \quad \sigma_T^{n+1} \cap \{x_1 = P\} = \emptyset;$$

$$\sigma_T^{n+2}\big|_{s=0} = \sigma_T^{n+2}\big|_{x_1=P}, \quad \sigma_T^{n+3}\big|_{s=0} = \sigma_T^{n+3}\big|_{x_1=P} \quad \text{for } t \in [0,T];$$

$$\sigma_T^{n+2}\big|_{s=s_{n+2}} = \sigma_T^{n+1}\big|_{s=0}, \quad \sigma_T^{n+3}\big|_{s=s_{n+3}} = \sigma_T^{n+1}\big|_{s=s_{n+1}} \quad \text{for } t \in [0,T].$$

We shall consider the case when in D_T there are n_1 nonclosed phase fronts

$$\sigma_T^i \equiv \{x_1 = x_1^i(s,t),\, x_2 = x_2^i(s,t)\},$$

where s is a parameter, $s \in [0, s_i]$, $i = 1, \dots, n_1$, $t \in [0, T]$, and moreover

$$\sigma_T^i \cap \sigma_T^{n+j}, \quad i = 1, 2, \dots, n_1, \quad j = 2, 3,$$

are smooth, continuous Jordan curves, and, in addition, there are n_2 closed phase fronts

$$\sigma_T^i \equiv \{x_1 = x_1^i(s,t),\, x_2 = x_2^i(s,t)\},$$

s is a parameter, $s \in [0, s_i]$, $t \in [0, T]$,

$$\sigma_T^i\big|_{s=0} = \sigma_T^i\big|_{s=s_i}, \quad i = n_1 + 1, \dots, n_1 + n_2,$$

such that

$$D_T^j \supset D_T^{j+1}, \quad j = n_1 + 1, \dots, n_1 + n_2 - 1;$$

$D_T^j(D_T^{j+1})$ is the domain* bounded by the closed front $\sigma_T^j(\sigma_T^{j+1})$, $j = n_1 + 1, \dots, n_1 + n_2 - 1$, and by the planes $t = 0$ and $t = T$; and, in addition, in D_T there are n_3 closed phase fronts

$$\sigma_T^i \equiv \{x_1 = x_1^i(s,t),\, x_2 = x_2^i(s,t)\},$$

s is a parameter, $s \in [0, s_i]$,

$$\sigma_T^i\big|_{s=0} = \sigma_T^i\big|_{s=s_i}, \quad i = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3,$$

such that

$$D_T^i \cap D_T^j = \emptyset \quad \text{for } i \neq j; \quad i, j = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3;$$

$D_T^i(D_T^j)$ is the domain bounded by the closed front $\sigma_T^i(\sigma_T^j)$ and by the planes $t = 0$ and $t = T$.

We denote: a) by D_T the domain bounded by the surfaces σ_T^{n+i}, $i = 1, 2, 3$, σ_T^1, and by the planes $t = 0$ and $t = T$; b) by D_T^k, $k = 1, 2, \dots, n_1 - 1$, the domain bounded by the surfaces σ_T^k, σ_T^{k+1}, σ_T^{n+2}, σ_T^{n+3}, and by the planes $t = 0$, $t = T$; c) by D_T^k, $k = n_1 + 1, \dots, n_1 + n_2 - 1$, the domain bounded by the surfaces σ_T^k, σ_T^{k+1}, and by the planes $t = 0$, $t = T$; d) by D_T^k, $k = n_1 + n_2, \dots, n_1 + n_2 + n_3$, the domain bounded by the surface σ_T^k and

* All domains, unless otherwise stated, are assumed to be closed.

planes $t = 0,\ t = T$; d) $D_T^{n_1} = D_T \setminus \bigcup_{i=1}^{n_1-1} D_T^i \setminus \bigcup_{i=n_1+1}^{n_1+n_2+n_3} D_T^i$.

Let $n_1 + n_2 + n_3 = n$ and $\sigma_T^i \cap \sigma_T^j = \emptyset$ for $i \neq j,\ i, j = 1, 2, \dots, n$; $D_T^k \cap D_T^j = \emptyset$ for $k = 1, 2, \dots, n_1 - 1,\ j = n_1 + 1, \dots, n$.

It is required to find $u^k(x_1, x_2, t),\ x_i^k(s,t),\ i = 1, 2;\ k = 1, \dots, n$, satisfying the conditions:

$$\sum_{i,j=1}^{2} a_{ij}^k(x_1,x_2,t)u_{x_ix_j}^k(x_1,x_2,t) + \sum_{i=1}^{2} b_i^k(x_1,x_2,t)u_{x_i}^k(x_1,x_2,t) +$$
$$+c^k(x_1,x_2,t)u^k(x_1,x_2,t) + F^k(x_1,x_2,t) = u_t^k(x_1,x_2,t)$$
$$\text{in } D_T^k, \quad k = 1,\dots,n, \tag{1}$$

$$\Lambda_0|\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}^k \xi_i \xi_j \le \Lambda_1 |\xi|^2, \qquad \Lambda_0, \Lambda_1 - \text{const} > 0,$$

$a_{ij}^k(x_1,x_2,t)$; $i,j = 1,2$, $b_i^k(x_1,x_2,t)$, $i = 1,2$, $c^k(x_1,x_2,t)$, $F^k(x_1,x_2,t)$, $k = 1,\dots,n$, are defined in D_T,

$$u^k(x_1,x_2,t)\big|_{t=0} = \varphi^k(x_1,x_2), \qquad (x_1,x_2) \in D_0^k, \quad k = 1,\dots,n; \tag{2}$$

$$u^k(x_1,x_2,t)\big|_{\sigma_T^{\gamma(k)}} = \omega_k^{\gamma(k)}(x_1^{\gamma(k)}(s,t), x_2^{\gamma(k)}(s,t), t) = \omega_k^{\gamma(k)}(s,t),$$

$$s \in [0, s_{\gamma(k)}],$$

$$t \in [0,T], \quad \gamma(k) = \{k, k+1 \text{ for } k = 1,2,\dots,$$
$$\dots, n_1 - 1, n_1 + 1, \dots, n_1 + n_2 - 1; \quad k \text{ for } n_1 + n_2, n_1 + n_2 + 1, \dots$$
$$\dots, n_1 + n_2 + n_3;\ n_1, n_1 + 1, n_1 + n_2 + 1, \dots, n + 1 \text{ for } k = n_1\}; \tag{3}$$

$$u^k(x_1,x_2,t)\big|_{\sigma_T^{n+i}} = \omega_k^{n+i}(x_1^{n+i}(s,t), x_2^{n+i}(s,t), t) = \omega_k^{n+i}(s,t),$$

$$t \in [0,T],\ s \in [s_k^{i+1}(t), s_{k+1}^{i+1}(t)],\ s_{n+1}^{i+1}(t) = s_{n+i+1}, \quad i = 1,2;\ k = 1,\dots,n_1; \tag{4}$$

$s_k^j(t)$ is determined from $\Phi^k(x_1^{n+j}(s,t), x_2^{n+j}(s,t), t) \equiv 0$, $k = 1,2,\dots,n_1$, $j = 2,3$, where $\Phi^k(x_1,x_2,t) = 0$ is the equation of the surface σ_T^k, $k = 1,2,\dots,n_1$, in Cartesian coordinates,

$$\partial x_i^k(s,t)/\partial t = \overline{\lambda_i^k}^{k-1}(s,t)u_{x_i}^{k-1}(x_1^k(s,t), x_2^k(s,t), t) -$$
$$-\lambda_i^{kk}(s,t)u_{x_i}^k(x_1^k(s,t), \lambda_2^k(s,t)),$$

$$\overline{\lambda_i^{kj}}(s,t) = \lambda_i^{kj}(s,t,x_1^k(s,t),x_2^k(s,t)), \quad u_{x_1}^0(x_1^1(s,t),x_2^1(s,t),t) \equiv 1,$$

$$i = 1,2, \quad j = k-1,k, \quad k = 1,2,\dots,n. \tag{5}$$

Definition 1. A regular classical solution of problem (1)–(5) in $D_{T'},\ 0 < T' \leq T$, is a system of functions $u^k(x_1,x_2,t),\ x_i^k(s,t),\ i = 1,2;\ k = 1,\dots,n$, satisfying the following conditions:

$$u^k(x_1,x_2,t),\ u_{x_i}^k(x_1,x_2,t),\ u_{x_ix_j}^k(x_1,x_2,t),\ u_t^k(x_1,x_2,t),$$

$$u_{x_it}^k(x_1,x_2,t),\ u_{x_ix_jx_\ell}^k(x_1,x_2,t)$$

are continuous in $D_T^k,\ k = 1,2,3,\dots,n$;

$$x_i^k(s,t),\ x_{is}^k(s,t),\ x_{iss}^k(s,t),\ x_{it}^k(s,t),\ x_{itt}^k(s,t),\ x_{ist}^k(s,t),\ i = 1,2;\ k = 1,\dots,n,$$

are continuous for $s \in [0,s_k],\ t \in [0,T']$, satisfy relations (1)–(5) and

a) $[x_{1s}^k(s,t)]^2 + [x_{2s}^k(s,t)]^2 \neq 0$ for $t \in [0,T'],\ s \in [0,s_k],\ k = 1,\dots,n$;

b) $\sigma_{T'}^i \cap \sigma_{T'}^j = \emptyset$ for $i \neq j;\ i,j = 1,2,\dots,n$;

c) $\sigma_{T'}^1 \cap \{x_1 = P\} = \emptyset$;

d)

$$\Phi_{x_1}^k(x_1^{n+j}(s,t),x_2^{n+j}(s,t),t)x_{1s}^{n+j}(s,t) + \Phi_{x_2}^k(x_1^{n+j}(s,t),$$

$$x_2^{n+j}(s,t),t)x_{2s}^{n+j}(s,t) \neq 0, \quad k = 1,2,\dots,n_1;\ j = 2,3;\ t \in [0,T'],\ s \in [0,s_k].$$

Theorem 1. Let: 1) $\varphi^k(x_1,x_2)\omega_k^{\gamma(k)}(x_1,x_2,t),\ \omega_k^{n+i}(x_1,x_2,t),\ i = 1,2,3;$ $a_{ij}^k(x_1,x_2,t),\ i,j = 1,2;\ k = 1,\dots,n$, be three times continuously differentiable; 2) $\psi_i^k(s),\ \lambda_i^{kj}(s,t),\ F_k,\ j = k,k-1,\ x_i^{n+j}(s,t),\ j = 1,2,3,\ b_i^k(x_1,x_2,t),\ i = 1,2,$ $k = 1,\dots,n$, be twice continuously differentiable; 3) $c^k(x_1,x_2,t)$ be continuously differentiable; 4)

$$[\psi_{1s}^k(s)]^2 + [\psi_{2s}^k(s)]^2 \geq \eta > 0;$$

5)

$$f_{x_1}^k(\psi_1^{n+j}(s),\psi_2^{n+j}(s))\psi_{1s}^{n+j}(s) + f_{x_2}^k(\psi_1^{n+j}(s),\psi_2^{n+j}(s))\psi_{2s}^{n+j}(s) \geq \eta' > 0, \quad j = 2,3,$$

where $s = s_k^j(0)$, $f^k(x_1, x_2) = 0$ is the equation of the curve $x_1 = \psi_1^k(s)$, $x_2 = \psi_2^k(s)$, $k = 1, 2, \dots, n_1$, in Cartesian coordinates; 6)

$$\min_{s \in [0, s_1]} x_1^1(s, t) > P,$$

the conjugacy conditions are fulfilled:

$$\varphi^k(\psi_1^{\mu(k)}(s), \psi_2^{\mu(k)}(s)) = \omega_k^{\mu(k)}(\psi_1^{\mu(k)}(s), \psi_2^{\mu(k)}(s), 0), \quad k = 1, 2, \dots, n,$$

$$\varphi_{x_m}^k(\psi_1^{\mu(k)}(s), \psi_2^{\mu(k)}(s)) = \omega_{kx_m}^{\mu(k)}(\psi_1^{\mu(k)}(s), \psi_2^{\mu(k)}(s), 0), \quad k = 1, 2, \dots, n; \ m = 1, 2,$$

$$\omega_k^i(x_1^i[(j-2)s_i, t], x_2^i[(j-2)s_i, t], t) = \omega_k^{n+j}(x_1^{n+j}(s_i^j(t), t), x_2^{n+j}(s_i^j(t), t), t),$$

$$\omega_{kx_m}^i(x_1^i[(j-2)s_i, t], x_2^i[(j-2)s_i, t], t) = \omega_{kx_m}^{n+j}(x_1^{n+j}(s_i^j(t), t), x_2^{n+j}(s_i^j(t), t), t),$$

$$\omega_{kt}^i(x_1^i[(j-2)s_i, t], x_2^i[(j-2)s_i, t], t) = \omega_{kt}^{n+j}(x_1^{n+j}(s_i^j(t), t), x_2^{n+j}(s_i^j(t), t), t),$$

$$k = 1, 2, \dots, n_1; \quad i = k, k+1 \text{ for } k \neq n_1; \quad i = n_1, n+1 \text{ for } k = n_1; \quad j = 2, 3,$$

$$\left[\sum_{i,j=1}^{2} a_{ij}^k \varphi_{x_i x_j}^k(x_1, x_2) + \sum_{i=1}^{2} b_i^k \varphi_{x_i}^k(x_1, x_2) + c^k \varphi^k(x_1, x_2) + \sum_{i=1}^{2} (\varphi_{x_i}^k - \omega_k^{\mu(k)}) x_{it}^{\mu(k)}(s, 0) = \omega_{kt}^{\mu(k)}(s, 0) \right]_{x_1 = \psi_1^{\mu(k)}(s),}$$

$$k = 1, 2, \dots, n;$$

$$\mu(k) = [k, k+1, n+2, n+3 \text{ for } k = 1, 2, \dots, n_1 - 1;$$

$$k, k+1 \text{ for } k = n_1 + 1, \dots, n_1 + n_2 - 1; \quad k \text{ for } k = n_1 + n_2, \dots, n;$$

$$n_1, n_1 + 1, n_1 + n_2 + 1, \dots, n + 3 \text{ for } k = n_1].$$

Then any solution of problem (1)–(5) satisfies, by virtue of (7), a system of Volterra integral equations of the second kind:

$$v = V(v, x, x_s), \qquad v_t = U(v, v_t, g, x, x_s, x_{ss}),$$

$$g = G(v, g, v_t, x, x_s, x_{ss}), \qquad x = X(v, x), \qquad x_s = Y(v, g, x, x_s), \tag{6}$$

$$x_{ss} = Z(v, g, v_t, x, x_s, x_{ss}),$$

where

$$v_m^{kl}(s,t) = u_{x_m}^k(x_1^l(s,t), x_2^l(s,t)), \quad v_{mt}^{kl}(st) = u_{x_{mt}}^k(x_1^l(s,t), x_2^l(s,t), t),$$

$$m = 1,2, \quad g_{mr}^{kl}(s,t) = u_{x_m x_r}^k(x_1^l(s,t), x_2^l(s,t), t), \quad m,r = 1,2; \quad k = 1,2,\dots,n$$

(l takes the same values as $\mu(k)$).

System (6) is written in symbolic form: the equations entering it are obtained from

$$u^k(x_1,x_2,t) = \iint_{D_0^k} \varphi^k(\xi_1,\xi_2)\Gamma_k \, d\xi_1 \, d\xi_2 + \sum_{\mu(k)} \iint_{\sigma_t^{\mu(k)}} \sum_{i=1}^{2} \left[\sum_{j=1}^{2} \left(a_{ij}^k v_j^{k\mu(k)} \Gamma_k - \omega_k^{\mu(k)} a_{ij}^k \Gamma_{k\xi_j} - \omega_k^{\mu(k)} a_{ij\xi_j}^k \Gamma_k \right) + \omega_k^{\mu(k)}\right.$$

[
{ (k)} { _t^{ (k)}} _k^{ (k)}(s,t) _k,d _1,d _2
+
_{D_t^k}F_k _k,d _1,d _2,d ;
k=1,...,n, m(i)=
]

$$= i - (-1)^i, \qquad i = 1,2, \tag{7}$$

and also from (5) and the remaining conditions of the problem. Here Γ_k is the fundamental solution of equation (1).

Definition 2. A **regular classical solution of the system of integral equations** (6) for $t \in [0, T)$, $s \in [0, s_k]$, $k = 1, 2, \ldots, n+3$, is a system of functions

$$x_i^k(s,t), \quad x_{is}^k(st), \quad x_{iss}^k(s,t), \quad i = 1,2; \quad k = 1,\ldots,n,$$

$$v_i^{kl}(s,t), \quad v_{it}^{kl}(s,t), \quad g_{mr}^{kl}(s,t),$$

$i, m, r = 1, 2$; $k = 1, \ldots, n$ (l takes the same values as $\mu(k)$), satisfying the conditions:
a) $x_i^k(s,t)$, $x_{is}^k(s,t)$, $x_{iss}^k(s,t)$, $i = 1,2$; $k = 1,2,\ldots,n$, are continuous for $s \in [0, s_k]$, $t \in [0, T']$, $k = 1, 2, \ldots, n$;
b) $v_i^{kl}(s,t)$, $g_{mr}^{kl}(s,t)$, $v_{it}^{kl}(s,t)$, $i, m, r = 1, 2$; $k = 1, \ldots, n$ (l takes the same values as $\mu(k)$) are continuous for $s \in [0, s_k]$, $k = 1, 2, \ldots, n+3$, $t \in [0, T']$, and satisfy relations (6) and relations a), b), c), d) of Definition 1.

Theorem 2. *Under the conditions of Theorem 1 there exists a T', $0 < T' \le T$, such that for the solution of the system of integral equations (6) the relations a), b), c), d) of Definition 1 are fulfilled.*

Theorem 3. *Suppose that all the conditions of Theorem 1 are fulfilled; then any solution of the system of integral equations (6) in the sense of Definition 2 gives, by virtue of (7), a solution of problem (1)–(5) in the sense of Definition 1.*

Let $B_{T'}$ be the Banach space of systems of functions continuous for $t \in [0, T']$, $s \in [0, s_k]$,

$$x_i^k(s,t), \quad x_{is}^k(s,t), \quad x_{iss}^k(s,t), \quad v_i^{kl}(s,t), \quad v_{it}^{kl}(s,t), \quad g_{mr}^{kl}(s,t), \qquad (8)$$

$i, m, r = 1, 2$; $k = 1, 2, \ldots, n$ (l takes the same values as $\mu(k)$) with a norm of type C.

By $B_{T'}^M$ we denote the set of systems of functions (8) from $B_{T'}$ for $t \in [0, T']$, $s \in [0, s_k]$, $k = 1, \ldots, n$, with norm (9) not exceeding the constant M. One considers the mapping $Q_{T''}$ of the set $B_{T''}^M$ by means of the system of integral operators standing in the right-hand sides of (6). It is proved that, under the conditions of Theorem 1, there exists a T'', $0 < T'' \le T'$, for which $Q_{T''}$ is a mapping of $B_{T''}^M$ into $B_{T''}^M$. If, in addition to the conditions of Theorem 1, the Lipschitz condition is also fulfilled for

$$a_{ijx_{mx}qx^r}^k, \quad b_{ix_{mx}r}^k, \quad c_{x_m}^k, \quad i = 1,2, \quad k = 1,\ldots,n; \quad m,q,r = 1,2, \quad x_0 = t,$$

$$\omega_{kx_{mx}qx^r}^{\mu(k)}, \quad F_{x_{mx}r}^k, \quad k = 1,\ldots,n, \quad \lambda_{ix_{mx}r}^{k\bar{j}}, \quad i = 1,2, \quad k = 1,\ldots,n, \quad j = k-1, k,$$

then the indicated T'' can be chosen so that $Q_{T''}$ will be a contraction mapping of $B_{T''}^M$ into $B_{T''}^M$; hence, by virtue of Theorem 2, the existence and uniqueness of a regular classical solution of system (6) is established, and consequently, by virtue of Theorem 3, the existence and uniqueness of a regular classical solution of problem (1)–(5). The stability of the solution of problem (1)–(5) with respect to perturbations of the coefficients and of the initial data is also proved.

Moscow State University
named after M. V. Lomonosov

Received
15 VII 1969

REFERENCES

1. S. L. Kamenomostskaya, *Matem. sborn.*, **33**, No. 4, 489 (1961).
2. A. Friedman, *Partial Differential Equations of Parabolic Type*, Moscow, 1968.
3. A. N. Tikhonov, *Bull. Moscow State Univ.*, No. 8 (1938).
4. B. M. Budak, M. Z. Moskal' , *DAN*, **184**, No. 6 (1969).
5. M. Gevrey, *J. Math. pures et appl.*, 6-s., IX, fasc. IV (1913).
6. V. P. Mikhailov, *Matem. sborn.*, **61**, No. 1 (1963); **62**, No. 2 (1964).
7. V. A. Solonnikov, *Tr. Mat. Inst. im. V. A. Steklova AN SSSR*, **83**, 5 (1966).
8. L. Rubinshtein, *The Stefan Problem*, Riga, 1967.

Note: Figure translations are in progress. See original paper for figures.