ON ONE GENERAL PROBLEM FROM INTERPOLATION THEORY

MATHEMATICS

1970

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Abstract

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UDC 517.949.12

MATHEMATICS

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ON ONE GENERAL PROBLEM FROM INTERPOLATION THEORY *

(Presented by Academician A. N. Kolmogorov on 24 III 1970)

Let $A(D)$ denote the space of functions analytic in an open simply connected domain D, lying in the finite part of the complex plane. The topology in this space is defined by uniform convergence on an arbitrary compact set $K \subset D$. $A^*(D)$ is the space of functions analytic outside the domain D and tending to zero at infinity. The topology in $A^*(D)$ is defined by uniform convergence on one of the closed sets F_α, $\forall\alpha$ $(F_\alpha \supset CD)$, $\prod_\alpha F_\alpha = CD$ (CD is the complement of the set D in the extended complex plane). It is well known that the spaces $A(D)$ and $A^*(D)$ are mutually adjoint.

Let the function $W(z) \in A(D)$ and be univalent in D. In addition, suppose that p functions $A_s(z)$, $s = 0, 1, \dots, p-1$, analytic in D, are given, $\forall A_s(z) \neq 0$, or p sequences of complex numbers $\{a_{ns}\}$, $n = 0, 1, 2, \dots$, $s = 0, 1, \dots, p-1$. Finally, fix p integers l_s, $s = 0, 1, \dots, p-1$, such that $0 \le l_s \le p-1$. It is not excluded that the l_s with different indices coincide.

An interpolation problem is posed, consisting of two parts.

I. Does there exist a function $\gamma(z) \in A^*(D)$ for which the equalities

$$\frac{1}{2\pi i}\int_\Gamma [W(z)]^{pn+l_s} A_s(z)\gamma(z)\,dz = a_{ns}, \qquad n = 0, 1, 2, \dots;\ s = 0, 1, \dots, p-1. \quad (1)$$

hold?

For a number of reasons, in (1) we do not combine the function $[W(z)]^{l_s}$ with $A_s(z)$, instead singling out $[W(z)]^{l_s}$ as a separate factor. The integration in (1) is carried out over a closed contour $\Gamma \subset D$, chosen so that the function $\gamma(z) \in A^*(D)$ is still regular in the closed unbounded domain $\overline{\Gamma} \cup \operatorname{ext}\Gamma \supset CD$ (in other words, $\gamma(z) \in A^*(\operatorname{int}\Gamma)$). Here $\operatorname{int}\Gamma$ and $\operatorname{ext}\Gamma$ denote, respectively, the "interior" and "exterior" domains into which Γ divides the complex plane.

II. If the collection of numerical sequences $\{a_{ns}\}$, $n = 0, 1, 2, \dots$, $s = 0, 1, \dots, p-1$, is such that there exists a function $\gamma(z) \in A^*(D)$ satisfying relations (1), then the problem arises of finding the entire set of functions $\gamma(z) \in A^*(D)$ satisfying the equalities (1).

This work is devoted to the solution of questions I and II.

It should be noted that the functionals (1) may be regarded not only as functionals on $A^*(D)$, but also as functionals on the space $[1; D)$ of entire functions of exponential type (and on any other space isomorphic to $A^*(D)$). Recall that $[1; D)$ is the space of entire functions of exponential type

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n,$$

for which the Borel-associated functions

$$\gamma(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{\zeta^{n+1}} \in A^*(D).$$

* The article is a brief exposition of part of a lecture delivered by me on August 27, 1969, at the International Colloquium on Function Theory in Budapest.

Problems I and II (their union we call the **general moment problem** (1)) contain, as special cases, a large number of widely known interpolation problems. For $p = 1$ and $A_0(z) \equiv 1$, this is the well-known moment problem of A. O. Gelfond (^{1,2}), while for $p = 1$, $A_0(z) = 1 - cz$ and $W(z) = ze^{-cz}$ it is the problem of Brouwé (^{3,4}). The general moment problem (1) reduces to the problems of reconstructing functions

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} e^{zt} \gamma(t)\, dt \in A(1; D)$$

from prescribed values of the derivatives $F^{(pn+l_s)}(\alpha_s)$, $n = 0, 1, 2, \dots,$ $s = 0, 1, \dots, p-1$, or $F^{(s)}(n)$, $n = 0, 1, 2, \dots,$ $s = 0, 1, \dots, p-1$, etc. Various particular cases of the listed problems have previously been investigated by many well-known mathematicians (see, in this connection, for example, ($^{1-12}$)). Therefore the formulation of the general moment problem (1) is fully justified.

We have proposed a new unified method for solving the general moment problem (1) under certain (though natural) restrictions on the function $W(z)$. This method makes it possible to solve questions I and II exhaustively for an extensive class of interpolation problems reducible to (1).

Let the function $w = W(z)$ map the domain D onto some domain G in the complex w-plane, and let the function $z = Z(w)$ be its inverse. It is assumed here that the domain G contains the origin and is invariant under rotation through the angle $\omega = 2\pi/p$ about the origin. Domains possessing this property will henceforth be called $2\pi/p$-invariant. The equalities (1) can be rewritten in the following equivalent form:

$$\frac{1}{2\pi i}\int_{\Gamma_w} w^{pn+l_s} A_s(Z(w))Z'(w)\gamma(Z(w))\,dw = a_{ns}. \tag{2}$$

$$(n = 0, 1, 2, \dots; \quad s = 0, 1, \dots, p-1).$$

In (2), Γ_w denotes the image of the contour Γ under the mapping $w = W(z)$. Moreover, we assume that the contour Γ has been chosen "sufficiently close" to the boundary ∂D and so that its image Γ_w is also invariant under rotation about the origin through the angle $\omega = 2\pi/p$.

Consider the functions

$$g_s(w) = \sum_{n=0}^{\infty} \frac{a_{ns}}{w^{pn+l_s+1}}, \qquad s = 0, 1, \dots, p-1,$$

whose Laurent coefficients $\{a_{ns}\}$ are the numbers on the right-hand sides of equality (1). From the relations (2) it follows that the functions $g_s(w)$, $s = 0, 1, \dots, p-1$, must belong to $A^*(\operatorname{int}\Gamma_w)$. The equalities (2) are equivalent to the fulfillment on the contour Γ_w of the following conditions:

$$\sum_{k=0}^{p-1} \delta^{k(l_s+1)} A_s\big(Z(w\delta^k)\big)\, Z'(w\delta^k)\gamma(Z(w\delta^k)) - pg_s(w) = \sum_{k=0}^{p-1} \delta^{k(l_s+1)} \Phi_s^+(w\delta^k), \tag{3}$$

$$s = 0, 1, \dots, p-1.$$

In (3), $\delta = e^{2\pi i/p}$, and $\Phi_s^+(w)$, $s = 0, 1, \dots, p-1$, are certain functions from $A(G)$. In passing from (2) to (3), the property of $2\pi/p$-invariance of the contour Γ_w was used. The boundary relations (3) lead to the consideration of the following rather peculiar boundary-value problem:

Find a function $\Psi^-(w) \in A^*(\operatorname{int}\Gamma_w)$ and p functions $\Psi_s^+(w) \in A(G)$, $s = 0, 1, \dots, p-1$, for which on the contour Γ_w the equalities

$$\sum_{k=0}^{p-1} \delta^{k(l_s+1)} A_s\big(Z(w\delta^k)\big) Z'(w\delta^k)\Psi^-(w\delta^k) - pg_s(w) = \Psi_s^+(w)$$

$$(s = 0, 1, \dots, p-1)$$

hold.

We have shown that this problem, for arbitrary $g_s(w) \in A^*(\operatorname{int}\Gamma_w)$, has a solution $\Psi^-(w) \in A^*(\operatorname{int}\Gamma_w)$ (and it is precisely this solution that is of primary interest to us!) under the condition that the determinant

$$\Delta(w) = \det \left\| \delta^{k(l_s+1)} A_s(Z(w\delta^k)) Z'(w\delta^k) \right\|_{s,k=0}^{p-1} \neq 0. \tag{4}$$

This solution $\Psi^-(w)$ is found by quadratures. Then the function

$$\gamma_1(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\Psi^-(W(t))}{t-z}\, dt, \qquad z \in \operatorname{ext}\Gamma, \tag{5}$$

is a solution of the interpolation problem (1). Thus an answer to question I has been obtained; this answer is contained in the following proposition.

Theorem 1. *A necessary and sufficient condition for solvability of the general moment problem (1) is the fulfillment of the relations*

$$g_s(w) = \sum_{n=0}^{\infty} \frac{a_{ns}}{w^{pn+l_s+1}} \in A^*(G), \qquad s = 0, 1, \dots, p-1.$$

Question II, concerning the recovery of all $\gamma(z) \in A^*(D)$ from the moments (1), is solved according to the following scheme. It is clear that the general solution of the interpolation problem (1) is the sum of a particular solution $\gamma_1(z)$—(5)—of this problem and the general solution $\gamma_0(z)$ of the homogeneous moment problem (1) (that is, when $\{a_{ns}\} \equiv 0$). The general solution of the homogeneous problem (1) depends on the number of zeros of the determinant $\Delta(w)$—(4)—that fall inside the domain $\Gamma_w \cup \operatorname{int}\Gamma_w$. Therefore, problems I and II for this case require some refinement. Namely, they should be posed and solved for a fixed contour Γ. Thus the refined problem (1) will be called the Γ-moment problem (1). The Γ-problem (1) is solved exhaustively by quadratures, using the apparatus of boundary-value problems in the theory of analytic functions, and the general solution $\gamma_0(z)$ of the homogeneous Γ-problem (1) is written in explicit form; this is a set of rational fractions of a very special class.

The method used in solving the Γ-moment problem employs the theory of boundary-value problems and is much stronger than is required by the formulation of the problem. But, on the other hand, this makes it possible to solve the Γ-problem in various other formulations as well. For example, one can formulate and solve the Γ-problem for functions participating in the equalities (1) that are regular respectively in the domains $\operatorname{int}\Gamma$ and $\exp\Gamma$, and merely continuous on the contour Γ itself. (On this point see, for example, (^{13}), where, among other things, a detailed bibliography of results on boundary-value problems related to what has been said is also given.)

If the domain G does not possess the property of $2\pi/p$-invariance, then the method proposed here does not allow one to solve the general moment problem (1). Naturally the question arises: is it the method of solution that fails in the situation that has arisen, or does the nature of the moment problem itself become more complicated? It turns out that the issue is not the method, but the complication of the problem itself; the uniqueness theorem stated below is to some extent convincing in this respect.

Let the domain D be the curvilinear quadrilateral P_h,

$$P_h = \{z : \ a < x = \operatorname{Re} z < b; \ \varphi(x) < y = \operatorname{Im} z < \varphi(x) + 2\pi(p-1) + h\},$$

where $h \in (0; 2\pi]$, and $\varphi(x) \in C[a; b]$, and suppose that the functionals (1) are given by the relations

$$L_{ns}(\gamma) = \frac{1}{2\pi i} \int_\Gamma e^{nz} A_s(z) \gamma(z)\, dz, \qquad n = 0, 1, 2, \dots; \quad s = 0, 1, \dots, p-1, \quad (6)$$

where $\gamma(z) \in A^*(P_h)$, and the functions $A_s(z) \in A(P_h)$, $s \neq 0, 1, \dots, p-1$. The contour Γ in (6) is chosen so that the closed simply connected domain $\Gamma \cup \operatorname{int} \Gamma \subset P_h$ and $\gamma(z) \in A^*(\operatorname{int} \Gamma)$.

Theorem 2. In order that from the equalities

$$L_{ns}(\gamma) = \frac{1}{2\pi i} \int_\Gamma e^{nz} A_s(z) \gamma(z)\, dz = 0, \quad n = 0, 1, 2, \dots, \quad s = 0, 1, \dots, p-1,$$

for an A-function $\gamma(z) \in A^*(P_h)$, $0 < h \leq 2\pi$, it follow that $\gamma(z) \equiv 0$, it is necessary and sufficient that the following conditions hold simultaneously:

1) the determinant $\det \|A_s(z + 2\pi i k)\|_{s,k=0}^{p-1}$ not vanish in the curvilinear quadrilateral

$$Q_1 = \{z : \ a < x < b; \ \varphi(z) < y < \varphi(x) + h\};$$

2) the minors $\mathfrak{A}_s(z)$, $s = 0, 1, \dots, p-1$, of the last column (corresponding to the value of the index $k = p-1$) of the determinant $\det \|A_s(z+2\pi i k)\|_{s,k=0}^{p-1}$ have no zeros common to all s, $s = 0, 1, \dots, p-1$, in the curvilinear quadrilateral

$$Q_2 = \{z : \ a < x < b; \ \varphi(x) + h \leq y \leq \varphi(x) + 2\pi\}.$$

For $h = 2\pi$, Q_2 should be understood as the curve $y = \varphi(x) + 2\pi$, $a < x < b$. It is easy to show that for $h > 2\pi$ there is no uniqueness for any $A_s(z)$, $s = 0, 1, \dots, p-1$, while for $h \leq 0$ Theorem 2 is false.

Let us explain that the functionals (6) reduce to (1) if in (6) one puts $W(z) = e^{z/p}$. The function $W(z) = e^{z/p}$ gives a conformal mapping of P_h onto the $2\pi/p$-invariant domain G–the annulus $e^{a/p} < |w| < e^{b/p}$, from which the "smaller part" enclosed between the curves

$$w = \exp\frac{x+i\varphi(x)}{p} \quad \text{and} \quad w = \exp\frac{x+i[\varphi(x)+2\pi(p-1)+h]}{p}, \qquad a \le x \le b,$$

has been removed; and the determinant appearing in Theorem 2 is nothing other than $\Delta(W(z))$–(4). Theorem 2 contains, as special cases, the results of the works ($^{10-12}$).

For the case of $2\pi/p$-invariance of the domain G, a necessary and sufficient condition for uniqueness of the solution of the general Γ-moment problem (1) is the absence of zeros of the function $\Delta(W(z))$ in the domain $\Gamma \cup \operatorname{int}\Gamma$ (with the possible exception only of a zero of prescribed multiplicity of the determinant $\Delta(W(z))$ that is the preimage of the origin under the mapping $w = W(z)$). Theorem 2 says that the uniqueness conditions for the solution of the general moment problem, generally speaking, become substantially more complicated in the absence of the $2\pi/p$-invariance property of the domain G.

In conclusion, I consider it a pleasant duty to express my gratitude to Prof. A. F. Leont' ev for his attention to this work.

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Received
19 III 1970

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Note: Figure translations are in progress. See original paper for figures.

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