ON THE UNIQUENESS OF DETERMINING THE SHAPE AND DENSITY OF A BODY IN INVERSE PROBLEMS OF POTENTIAL THEORY

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.944

MATHEMATICS

A. I. PRILEPKO

ON THE UNIQUENESS OF DETERMINING THE SHAPE AND DENSITY OF A BODY IN INVERSE PROBLEMS OF POTENTIAL THEORY

(Presented by Academician M. A. Lavrent' ev on January 4, 1970)

The main content of this work is devoted to the investigation of uniqueness of the solution of mixed inverse problems of potential theory, namely, problems of determining the shape and density of a body from known values of the potential of volume masses or of the simple-layer potential. Various investigations in this direction are carried out in the work.

First, we consider a mixed problem of determining the shape and density of a body from the values of the exterior potential (see ([3,11,7])), which includes the problem of determining the shape of a body of given density from the exterior potential (see ([4–6,8–10])), as well as the problem of determining the density of a given body from the exterior potential (see ([8])). In the present paper, for the mixed problem, uniqueness theorems are proved for various classes of "contact" bodies (star-shaped with respect to an exterior finite or infinite point) and various classes of variable densities.

Second, for the case when the mixed exterior problem has, in general, a nonunique solution, the mutual arrangement of the domains is clarified depending on the arrangement of the densities. For example, if the exterior Newtonian potentials are equal, then the problem arises of changing the domain under a change of constant density (see ([4,12,1])). In this work the second case is investigated for various classes of bodies of variable densities.

Third, for the case when the mixed problem has a nonunique solution, the question arises of finding minimal additional requirements on the potentials that ensure uniqueness of the determination of the shape and density of the body. Such an additional condition in the paper is equality of the potentials at least at one interior point belonging to the intersection of the bodies (see also ([1]), where the additional condition is equality of the exterior potentials of bodies with a density having an additional known factor).

1°. Let T_α $(\alpha = 1, 2)$ be finite domains of Euclidean space E^n $(n \geq 2)$ with piecewise smooth (smoothness of class $A^{(1,\lambda)}$) boundaries S_α. Denote by $U^\alpha(x) = U(x; T_\alpha, \mu_\alpha)$ the generalized potential of volume masses $(n \geq 2)$ with density $\mu_\alpha \neq 0$ almost everywhere for points of the domain T_α. Denote by $U^\alpha(x) = U(x; S_\alpha, \nu_\alpha)$ the generalized simple-layer potential with density $\nu_\alpha \neq 0$ almost everywhere for points of the boundary S_α (see ([2,8])).

If the elliptic equation of second order has the form

$$\Delta u - \chi^2 u = 0, \qquad \chi = \text{const} \geq 0,$$

then for $\chi \geq 0$ the generalized potentials will be called metaharmonic, and if $\chi = 0$, classical.

For the investigation of uniqueness, we give the formulation of the general mixed exterior problem of potential theory.

Problem 1. Find conditions on the domains T_α and densities μ_α $(\alpha = 1, 2)$ such that the requirement of equality of the exterior potentials,

$$U(x; T_1, \mu_1) = U(x; T_2, \mu_2) \quad \text{for } x \in E^n \setminus (\overline{T}_1 \cup \overline{T}_2), \tag{1}$$

there followed the equality

$$T_1 = T_2, \qquad \mu_1 = \mu_2. \tag{2}$$

If in the conditions of problem 1 we set $\mu_1 = \mu_2$ or $T_1 = T_2$, then we obtain, respectively, problem 1_1 on determining the shape of a body from the values of the exterior volume potential, or problem 1_2 on determining the density of a given body from the values of the exterior volume potential.

We first consider the mixed inverse problem for classical volume potentials $(n \geqslant 2)$.

Let Γ be a given surface of class $A^{(1,\lambda)}$; we shall say that Γ is a **common contact surface for the domains** T_α $(\alpha = 1, 2)$, if a part of each of the boundaries S_α $(\alpha = 1, 2)$ is situated on this surface.

Condition I. Let the simply connected domains T_α belong to the ring K:

$$K = \{y;\ \rho_0 < |y| < R\}, \quad 0 < \rho_0 < R; \quad \rho_0,\ R = \text{const}.$$

Suppose that the contact surface Γ belongs to K, and that Γ is star-shaped with respect to the origin O. It is additionally assumed that, for each $\alpha = 1, 2$, the noncontact part $S_\alpha \setminus \Gamma$ of the boundary of the domain T_α is star-shaped with respect to the point O.

Theorem 1. Let the domains T_α, having the common contact surface Γ, satisfy condition I. Suppose that the functions $\mu_\alpha(y)$ of class $C^1(\overline{K})$ satisfy the condition

$$\frac{\partial}{\partial \rho}(\rho^n \mu_\alpha) = 0, \qquad \rho = |y|, \qquad y = (y_1, \dots, y_n). \tag{3}$$

If, for the volume potentials $U(x; T_\alpha, \mu_\alpha)$ of the indicated domains and densities, the equality

$$U(x; T_1, \mu_1) = U(x; T_2, \mu_2) \quad \text{for} \quad x \in E^n \setminus (\overline{T}_1 \cup \overline{T}_2), \tag{4}$$

holds, then

$$T_1 = T_2; \qquad \mu_1(y) = \mu_2(y), \qquad y \in T_1. \tag{5}$$

Proposition 1. Assertion (5) of Theorem 1 is also valid in the case when $S_\alpha \setminus \Gamma$ are situated in the direction of increasing ρ from Γ and all the conditions of Theorem 1 are fulfilled, while the densities $\mu_\alpha(y)$, instead of condition (3), satisfy the condition

$$\mu_1 \geqslant \mu_2 > 0, \qquad \partial \mu_\alpha / \partial \rho = 0. \tag{6}$$

From this proposition we obtain

Corollary 1. If $S_\alpha \setminus \Gamma$ are situated in the direction of increasing ρ from Γ and all the conditions of Theorem 1 are fulfilled, while the densities $\mu_\alpha(y) > 0$, instead of (3), satisfy the condition

$$\mu_\alpha = \delta_\alpha v(y), \qquad \text{where } \delta_\alpha = \text{const} > 0, \quad \partial v / \partial \rho = 0, \tag{7}$$

then equality (5) holds.

We present a limiting case of Theorem 1, when the center of star-shapedness O is at "infinity," i.e., if $y = (y_1, \dots, y_n)$ is a point of E^n $(n \geqslant 2)$, then the center of star-shapedness is at $\pm\infty$ with respect to the axis Oy_n.

Condition II. Let the simply connected finite domains T_α $(\alpha = 1, 2)$ have a common contact surface Γ, and let the surface Γ be such that every straight line parallel to the axis Oy_n intersects Γ in no more than one point (or segment).

Additionally suppose that, for each $\alpha = 1, 2$, the noncontact part $S_\alpha \setminus \Gamma$ of the boundary of the domain T_α has the property that any straight line parallel to the axis Oy_n intersects $S_\alpha \setminus \Gamma$ in no more than one point (or in one segment).

Theorem 2. Let the domains T_α ($\alpha = 1, 2$) satisfy condition II, and let the functions $\mu_\alpha > 0$ of class $C^1(\overline{T}_\alpha)$ satisfy the condition

$$\partial \mu_\alpha / \partial y_n = 0. \tag{8}$$

If, for the potentials $U(x; T_\alpha, \mu_\alpha)$ of the indicated domains and densities, the equality

$$U(x; T_1, \mu_1) = U(x; T_2, \mu_2) \quad \text{for } x \in E^n \setminus (\overline{T}_1 \cup \overline{T}_2),$$

holds, then

$$T_1 = T_2; \qquad \mu_1(y) = \mu_2(y), \quad y \in T_1.$$

Remark 1. Let us note that Theorems 1-2, in the case of problem 1_1 or problem 1_2, respectively, when the densities or the domains are prescribed, were proved earlier in the author' s papers ([8]) for broader classes of bodies and variable densities. In the general case, for the mixed problem 1, Theorem 2 for the two-dimensional case, under the condition of analyticity of the boundaries $S_\alpha \setminus \Gamma$, where Γ is the straight line $y_n = \text{const}$ and $\mu_\alpha = \text{const}$, was proved in ([3]) (see also ([11])). For variable densities of class (8), and T a plane $y_n = \text{const} = H$, under certain additional conditions on $S_\alpha \setminus \Gamma$, Theorem 2 was proved in ([7]).

Remark 2. Theorems 1, 2 and analogues of them are also valid for a broader class of bodies $A\alpha$, as well as for generalized potentials of certain elliptic equations of the second order.

2°. We note that if, in the hypotheses of Theorem 1, the existence of a common contact surface is not assumed, then, generally speaking, problem 1 has a nonunique solution, for example, in the case of spherical bodies with constant densities.

It turns out that the following holds.

Theorem 3. Let T_α be domains star-shaped with respect to a common interior point $0 \in T_1 \cap T_2$, and let the functions $\mu_\alpha(y)$ satisfy the condition

$$\mu_1 > \mu_2 > 0, \qquad \partial \mu_\alpha / \partial \rho = 0 \quad (\alpha = 1, 2). \tag{9}$$

If the exterior potentials of the given domains and densities coincide, i.e. if (1) holds, then

$$T_2 \supset \overline{T}_1.$$

Let us note that Theorem 3 generalizes the assertion $T_2 \supset T_1$ of ([12]), where this theorem was proved for the case $\mu_\alpha = \text{const}$.

Theorem 4. Let T_α be domains star-shaped with respect to a common point $O \in T_1 \cap T_2$, and let the functions μ_α satisfy condition (9). If the equality

$$U(x; T_1, \mu_1) = U(x; T_2, \mu_2) \quad \text{for } x \in E^n \setminus (\overline{T}_1 \cup \overline{T}_2)$$

holds and, moreover, there exists at least one point $x_0 \in T_1 \cap T_2$ such that

$$U(x_0; T_1, \mu_1) = U(x_0; T_2, \mu_2),$$

then

$$T_1 = T_2; \qquad \mu_1(y) = \mu_2(y), \quad y \in T_1.$$

Corollary 2. The assertion of Theorem 4 holds if all the conditions of the theorem are satisfied, and the densities μ_α, instead of (9), satisfy the condition

$$\mu_\alpha = \delta_\alpha \nu(y), \quad \text{where } \delta_\alpha = \text{const} > 0,\ \nu(y) > 0,\ \partial\nu/\partial\rho = 0.$$

3°. We present theorems, similar to Theorems 3-4, for generalized potentials of a simple layer.

Theorem 5. Let T_α be convex bodies, and let the functions $\nu_\alpha = \text{const}$ satisfy the condition

$$\nu_1 > \nu_2 > 0.$$

If the generalized simple-layer potentials $U(x; S_\alpha, \nu_\alpha)$ satisfy the condition

$$U(x; S_1, \nu_1) = U(x; S_2, \nu_2) \quad \text{for } x \in E^n \setminus (\overline{T}_1 \cup \overline{T}_2),$$

then

$$T_2 \supset \overline{T}_1.$$

Remark. If in the conditions of Theorem 5 we have $v_1 = v_2$, then $T_1 = T_2$ (see (8)); but if $v_1 > v_2$, then there exist examples of two spherical surfaces for which $T_2 \supset T_1$.

Theorem 6. Let T_α be convex bodies, $v_\alpha = \text{const}$. If the equality

$$U(x; S_1, v_1) = U(x; S_2, v_2) \quad \text{for } x \in E^n \setminus (\overline{T}_1 \cup \overline{T}_2)$$

holds and, in addition, there exists at least one point $x_0 \in T_1 \cap T_2$ such that

$$U_0(x_0; S_1, v_1) = V(x_0; S_2, v_2),$$

then

$$S_1 = S_2, \qquad v_1 = v_2.$$

The proof of Theorems 1-6 is carried out by the method of "generalized moments," developed by the author for the study of uniqueness of solutions of inverse problems (see (8)).

Institute of Mathematics Siberian Branch of the Academy of Sciences of the USSR Novosibirsk

Received 24 XII 1969

REFERENCES CITED

1. M. A. Atakhodzhaev, *Sibirsk. matem. zhurn.*, 7, No. 2, 455 (1966).

2. A. V. Bitsadze, *Boundary-value problems for elliptic equations of the second order*, Moscow, 1969.

3. A. A. Zamorev, *Izv. AN SSSR, ser. geograf. i geofiz.*, No. 1, 48 (1942).

4. V. K. Ivanov, *Izv. vyssh. uchebn. zaved., Matem.*, No. 3, 99 (1958).

5. M. M. Lavrent' ev, *On some ill-posed problems of mathematical physics*, Novosibirsk, 1962.

6. P. S. Novikov, *DAN*, 18, No. 3, 165 (1938).

7. A. Kh. Ostromogil' skii, *Zhurn. vychislit. matem. i matem. fiz.*, 9, No. 5, 1189 (1969).

8. A. I. Prilepko, *DAN*, 181, No. 5, 1065 (1968); *Differents. uravn.*, 2, No. 1, 107 (1966); No. 2, 194 (1966); 3, No. 1, 30 (1967); 4, No. 1, 71 (1968).

9. L. N. Sretenskii, *DAN*, 99, No. 1, 21 (1954).

10. A. N. Tikhonov, *DAN*, 39, No. 5, 195 (1943).

11. A. V. Tsirul' skii, *Izv. AN SSSR, ser. geofiz.*, No. 11, 1693 (1964).

12. Yu. A. Shapkin, *DAN*, 118, No. 1, 45 (1958).