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ON THE REGULARIZATION OF SINGULAR OPERATOR EXPRESSIONS IN QUANTUM FIELD THEORY

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Up to now, the only illustration of the general postulates of quantum field theory has been the divergent series of perturbation theory (^{1}). One of the approaches to constructing a nontrivial model using Fock space consists in violating the conditions of Haag' s theorem (^{2,3}). In this connection, regularization methods that violate Euclidean invariance acquire important significance. In this direction we propose a method for regularizing singular operator expressions as applied to φ^4-theory, as a development of an approach proposed earlier for generalizing annihilation and creation operators (see (^{4}), § 1).

I. Let us introduce the basic notation and definitions. Let $K(\mathbf{p},\mathbf{q})$ be a kernel satisfying the following requirements:

1) $K(\mathbf{p},\mathbf{q})$ is a continuous function of the variables $\mathbf{p},\mathbf{q}\in R^3$;
2) $K(\mathbf{p},\mathbf{q}) = K(\mathbf{q},\mathbf{p})$ for all $\mathbf{p},\mathbf{q}$;
3)
$$\iint f(\mathbf{p})K(\mathbf{p},\mathbf{q})f(\mathbf{q})\,d\mathbf{p}\,d\mathbf{q} \geq 0$$
for all $f(\mathbf{p}) \in L_2(R^3)$, and equality to zero implies $f(\mathbf{p}) \equiv 0$;
4)
$$\iint K(\mathbf{p},\mathbf{q})[\omega(\mathbf{p})\omega(\mathbf{q})]^{-1/2}\,d\mathbf{p}\,d\mathbf{q} < \infty, \quad \omega(\mathbf{p}) = \sqrt{\mathbf{p}^2+m^2}, \quad m>0.$$

These properties are satisfied, for example, by the kernel

$$K_s(\mathbf{p},\mathbf{q}) = \frac{1}{(\pi s)^{3/2}} \exp\left[-\frac{(\mathbf{p}-\mathbf{q})^2}{s} - s(\mathbf{p}^2+\mathbf{q}^2)\right], \quad 0<s<\infty,$$

which will be used essentially below.

We are now ready to introduce an analogue of the Fock space F_0. Denote by F_- the Hilbert space of sequences $G = \{G_0,\ G_1(\mathbf{p}_1), \dots, G_n(\mathbf{p}_1, \dots, \mathbf{p}_n), \dots\}$, whose scalar product has the form

$$(F, G)_- = \sum_{n=0}^{\infty} \iint d\mathbf{p}_1\, d\mathbf{p}'_1\, K(\mathbf{p}_1, \mathbf{p}'_1) \cdots \iint d\mathbf{p}_n\, d\mathbf{p}'_n\, K(\mathbf{p}_n, \mathbf{p}'_n) \times$$

$$\times F_n(\mathbf{p}_1, \dots, \mathbf{p}_n)\, \overline{G(\mathbf{p}'_1, \dots, \mathbf{p}'_n)}.$$

All functions $G_n(\mathbf{p}_1, \dots, \mathbf{p}_n)$, for arbitrary $n = 1, 2, \dots$, are assumed to be symmetric with respect to their arguments. Corresponding to the scalar product introduced,

$$F_- = \oplus \sum_{n=0}^{\infty} F^{(n)}(R^{3n}).$$

Let us note the following two important properties of the spaces F_0 and F_-. First, the embedding $F_0 \to F_-$ is quasinuclear. Second, among the functions $G_n(\mathbf{p}_1, \dots, \mathbf{p}_n)$ there are also δ-functions, i.e., there exists a function $\delta_{p_1, \dots, \mathbf{p}_n}$ for which

$$\left\| \delta_{p_1, \dots, \mathbf{p}_n} \right\|_{F_-^{(n)}} = K(\mathbf{p}_1, \mathbf{p}_1) \cdots K(\mathbf{p}_n, \mathbf{p}_n) > 0.$$

II. In the space F_- we introduce analogues of the Fock annihilation and creation operators $a^-(\mathbf{k})$, $a^+(\mathbf{k})$ as follows:

$$(A^-(F_1)G)_{n-1}(\mathbf{k}_1, \dots, \mathbf{k}_{n-1}) =$$

$$= \sqrt{n} \iint d\mathbf{k}_n\, d\mathbf{k}'_n K(\mathbf{k}_n, \mathbf{k}'_n) F_1(\mathbf{k}'_n) G_n(\mathbf{k}_1, \dots, \mathbf{k}_n),$$

$$(A^+(F_1)G)_{n+1} = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} G_n(\mathbf{k}_1, \dots, \hat{\mathbf{k}}_j, \dots, \mathbf{k}_{n+1}) F_1(\mathbf{k}_j),$$

where $F_1(\mathbf{k}_1)$ is any real function from $F_-^{(1)}(R^3)$. It is convenient to write these operators also in the following form

$$(A^-(\mathbf{k})G)_{n-1}(\mathbf{k}_1, \dots, \mathbf{k}_{n-1}) = \sqrt{n} \int d\mathbf{k}_n K(\mathbf{k}, \mathbf{k}_n) G_n(\mathbf{k}_1, \dots, \mathbf{k}_n),$$

$$(A^+(\mathbf{k})G)_{n+1}(\mathbf{k}_1, \dots, \mathbf{k}_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} G_n(\mathbf{k}_1, \dots, \hat{\mathbf{k}}_j, \dots, \mathbf{k}_{n+1}) \delta(\mathbf{k} - \mathbf{k}_j),$$

where $A^+(\mathbf{k})$ has the meaning of a generalized operator-valued function.

The following properties hold:

1) the operators $A^-(F_1)$ and $A^+(F_1)$ have a common dense domain of definition;
2) the operators $A^-(F_1)$ and $A^+(F_1)$ are mutually adjoint in the scalar product $(\cdot, \cdot)_-$, and $[A^-(F_1)]^* = A^+(F_1)$;
3) the commutation relations hold

$$[A^-(F_1), A^-(G_1)] = [A^+(F_1), A^+(G_1)] = 0,$$

$$[A^-(F_1), A^+(G_1)] = (F_1, G_1)_{F_-^{(1)}(R^3)}.$$

III. Introduce the operators

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\mathbf{k}x} \{A^+(\mathbf{k}) + A^-(-\mathbf{k})\} \frac{d\mathbf{k}}{\sqrt{2\omega(\mathbf{k})}}$$

in the space F_-, generated by the kernel $K_s(p, q)$.

The operator $\varphi(F_1)$, where F_1 is an arbitrary real function $F_1(\mathbf{k}) \in F_-(R^3)$, is self-adjoint in F_-.

Construct the operators

$$H_0 = \int \omega(\mathbf{k}) A^+(\mathbf{k}) A^-(\mathbf{k})\, d\mathbf{k}, \qquad H_{\text{int}} = \int :\varphi^4(x):\, dx =$$

$$= \sum_{j=0}^{4} \binom{4}{j} \int \cdots \int \frac{\delta(\mathbf{k}_1 + \cdots + \mathbf{k}_4)}{\omega(\mathbf{k}_1) \cdots \omega(\mathbf{k}_4)} A^+(\mathbf{k}_1) \cdots A^+(\mathbf{k}_j) A^-(-\mathbf{k}_{j+1}) \cdots A^-(-\mathbf{k}_4)\, d\mathbf{k}_1 \cdots d\mathbf{k}_4,$$

$$H = H_0 + \lambda H_{\text{int}}, \qquad \lambda > 0.$$

By obvious analogy, we shall call the expression H the regularized full Hamiltonian of φ^4-theory.

Theorem. The regularized full Hamiltonian H is a symmetric operator in the Hilbert space F_{-}, generated by the kernel $K_s(p,q)$.

The self-adjoint properties of the Hamiltonians H_0, H_{int}, and H will be studied in a subsequent work. Here we restrict ourselves to one remark, which will make it possible to understand the proposed regularization of the annihilation and creation operators from the point of view of regularizing singular functions in quantum field theory.

IV. In our scheme the usual positive-frequency function $D^{+}(x-y)$ is represented by the function

$$D_s^{+}(x,y)=\frac{i}{(2\pi)^3}\iint e^{ipx+iqy}\theta(p^0)\delta(p^2-m^2)\theta(q^0)\delta(q^2-m^2)K_s(p,q)\,dp\,dq.$$

since $D_s^{+}(x,y)$ is defined in terms of the operators $\varphi(\mathbf{x},x^0)=\varphi(x)$ in the following way:

$$\frac{1}{i}D_s^{+}(x,y)=(\psi_0,\varphi(x)\varphi(y)\psi_0)_{-},$$

where $\psi_0=\{1,0,0,...\}$ is a cyclic vector in F_{-} and $A^{-}(k)\psi_0=0$. Since the function $D_s^{+}(x,y)$ has no singularities on the light cone and is smooth, and since $K_s(\mathbf{p},\mathbf{q})\to\delta(\mathbf{p}-\mathbf{q})$, $s\to 0$ in the weak sense, the following is valid.

Lemma. The product

$$\prod_{(u<v)}^{N} D^{+}(x_u-y_v)$$

can be defined as the weak limit of regular functions

$$\prod_{(u<v)}^{N} D_s^{+}(x_u,y_v)$$

on the class of basic functions $S(R^{3N})$.

V. As follows from the explicit form of $D_s^{+}(x,y)$, the proposed regularization is essentially based on a violation of the translational invariance of the theory.

Nevertheless, the commutation relation

$$[\varphi(\mathbf{x},0),\varphi(\mathbf{y},0)]=0$$

holds in the Hilbert space F_{-}, since

$$[A^-(\mathbf{k}), A^+(\mathbf{k}')] = K_s(\mathbf{k}, \mathbf{k}').$$

Consequently, the constructed regularization approximates the canonical commutation relations by replacing $\delta(\mathbf{k} - \mathbf{k}')$ with the δ-like sequence $K_s(\mathbf{k}, \mathbf{k}')$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.