AN INTEGRAL-GEOMETRY PROBLEM CONNECTED WITH A PAIR OF GRASSMANN MANIFOLDS

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Abstract

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The aim of this work is the formulation and solution of an integral-geometry problem for a pair of manifolds—the manifold $G_{n,l}$ of l-dimensional oriented subspaces in R^n and the manifold $G_{n,k}$ of k-dimensional oriented subspaces in R^n. In what follows it is assumed that $l < k$, $l + k \leq n$, and also that $k - l$ is an even number*. The results of the work carry over without essential changes also to the case of complex Grassmann manifolds (Section 4).

1. Consider the manifold $E_{n,l}$ of all l-frames $v = (v_1, \dots, v_l)$ in R^n; endow it with the structure of a principal fiber space of the group $GL_0(l)$ with base $G_{n,l}$ ($GL_0(l)$ is the subgroup of linear transformations in R^l preserving orientation**). Analogously, let $E_{n,k}$ be the manifold of all k-frames $u = (u_1, \dots, u_k)$ in R^n, endowed with the structure of a principal fiber space of the group $GL_0(k)$ with base $G_{n,k}$. We introduce the following function spaces: 1) the space $S_k(E_{n,l})$ of infinitely differentiable functions $f(v)$ on $E_{n,l}$ satisfying the relation

$$f(v\nu) = |\det \nu|^{-k} f(v) \quad \text{for every } \nu \in GL(l), \tag{1}$$

and 2) the space $S_l(E_{n,k})$ of infinitely differentiable functions $\varphi(u)$ on $E_{n,k}$ satisfying the relation

$$\varphi(u\mu) = |\det \mu|^{-l} \varphi(u) \quad \text{for every } \mu \in GL(k)^{***}. \tag{2}$$

In the work a linear mapping $S_k(E_{n,l}) \to S_l(E_{n,k})$ is studied, which will now be defined.

Let $u = (u_1, \dots, u_k) \in E_{n,k}$, and let h be the oriented k-dimensional subspace spanned by u. Consider the manifold $E_{k,l}$ of all l-frames in R^n belonging to h. Since a basis $u = (u_1, \dots, u_k)$ is fixed in h, a system of coordinates is thereby defined in $E_{k,l}$: to each l-frame $v = (v_1, \dots, v_l)$ there corresponds the matrix

$\|t_j^i\|_{\substack{i=1,\dots,k\\ j=1,\dots,l}}$, where $(t_j^1, \dots, t_j^k)$ are the coordinates of v_j in the basis $u_1, \dots, u_k$ $(1 \le j \le l)$. Define on $E_{k,l}$ the following differential form of degree $N = (k-l)l$:

$$\sigma(t) = \sigma_1(t) \wedge \dots \wedge \sigma_l(t), \tag{3}$$

where

$$\sigma_i(t) = \sum_{(p_1,\dots,p_k)} (-1)^s t_1^{p_1} \cdots t_l^{p_l}\, dt_i^{p_{l+1}} \wedge \dots \wedge dt_i^{p_k} \tag{4}$$

(the summation is over all permutations $(p_1, \dots, p_k)$ of the indices $1, \dots, k$; s is the parity of the corresponding permutation).

* For the case $l = 1$, the integral-geometry problem was solved by somewhat different methods in ([2]). The case $k + l = n$ (the metric variant) was considered in ([3]).

** The projection $E_{n,l} \to G_{n,l}$ assigns to each l-frame v the l-dimensional oriented subspace spanned by it.

*** Functions from $S_k(E_{n,l})$ and $S_l(E_{n,k})$ can be interpreted as sections of one-dimensional vector bundles over $G_{n,l}$ and $G_{n,k}$, respectively.

Proposition 1. The differential form $\sigma(t)$, under transformations $t \mapsto \mu t \nu$, $\mu \in GL(k)$, $\nu \in GL(l)$, is multiplied by $(\det \mu)^l (\det \nu)^k$.

Proposition 2. If $f \in S_k(E_{n,l})$, then the differential form $f(ut)\sigma(t)$ on $E_{k,l}$ can be lowered from $E_{k,l}$ to the base $G_{k,l}$ of the natural fibration $E_{k,l} \to G_{k,l}$.

We define the mapping $S_k(E_{n,l}) \to S_l(E_{n,k})$ by the formula

$$\varphi(u) = \int_{G_{k,l}} f(ut)\sigma(t). \tag{5}$$

The validity for $\varphi(u)$ of relation (2) follows easily from Proposition 1.

The paper gives a description of the image of the mapping $S_k(E_{n,l}) \to S_l(E_{n,k})$ (Theorem 2) and obtains an inversion of formula (5) (Theorem 4).

2. Introduce the space F of all possible pairs $(h; \beta)$, where $h \in G_{n,k}$, $\beta = (\beta^{(1)}, \dots, \beta^{(l)})$ is an l-frame in the space h' conjugate to h. We shall endow the space F in a natural way with the structure of a manifold and of a fibered space over $G_{n,k}$ with fiber $E_{k,l}$.

To a function $\varphi(u) \in S_l(E_{n,k})$ we associate a differential form on F of degree $N = (k-l)l$. To this end introduce the following differential form on $E_{n,k}$:

$$\omega_N = \chi_N \varphi = \left(\bigwedge_{i=1}^{l} \bigwedge_{j=l+1}^{k} d_j^i \right) \varphi(u), \tag{6}$$

where

$$d_j^i = \sum_{s=1}^{n} \frac{\partial}{\partial u_j^s} \, du_j^s. \tag{7}$$

Define the fibration $\pi : E_{n,k} \to F$, which assigns to $(u_1, \dots, u_k) \in E_{n,k}$ the element $(h; \beta^{(1)}, \dots, \beta^{(l)}) \in F$, where h is the oriented k-dimensional subspace spanned by $u_1, \dots, u_k$, and $\beta^{(i)}$ $(1 \le i \le l)$ are vectors from h', determined by the relations: $(\beta^{(i)}, u_j) = \delta_j^i$ for $j \le l$, $(\beta^{(i)}, u_j) = 0$ for $j > l$.

Theorem 1. The differential form ω_N is the image of a differential form on F under the mapping π^*, induced by the mapping π; in other words, ω_N can be lowered from $E_{n,k}$ to F.*

Denote by $G_v \subset G_{n,k}$ the submanifold of subspaces $h \in G_{n,k}$ containing a given frame $v \in E_{n,l}$; let, further, $F_v \subset F$ be the totality of all $(h; \beta^{(1)}, \dots, \beta^{(l)})$ such that $h \in G_v$ and $(\beta^{(i)}, v_j) = \delta_j^i$, $i, j = 1, \dots, l$.

Theorem 2. In order that a function $\varphi(u) \in S_l(E_{n,k})$ be the image of some function $f(v) \in S_k(E_{n,l})$, i.e. be representable in the form (5), it is necessary and sufficient that the differential form $\omega_N = \chi_N \varphi$ be closed on each submanifold F_v.

3. To obtain an inversion formula, starting from the differential form ω_N, construct a differential form on G_v. For this purpose choose an arbitrary section $s : G_v \to F_v$. It induces an inverse mapping s^* of differential forms. Thus, $s^* \omega_N$ is a differential form on G_v.**

Theorem 3. Let two arbitrary sections $s_1 : G_v \to F_v$ and $s_2 : G_v \to F_v$ be given. Then, if the function $\varphi(u)$ is representable in the form (5), the difference $s_1^* \omega_N - s_2^* \omega_N$ is an exact differential form on G_v.

* In [2], where the case $l = 1$ was considered, the differential form was defined not on F, but on the manifold $L' \supset F$ of all pairs (h, β), where $h \in G_{n,k}$, and β is an arbitrary (not necessarily nonzero) vector from h'. Here L' was defined as the base of the principal fibration $E_{n,k} \times (\mathbb{R}^k)' \to L'$, and the form ω_N was defined through its lifting to $E_{n,k} \times (\mathbb{R}^k)'$. The method of specifying the differential form ω_N given here is technically more convenient.

** If $k + l > n$, then $\deg \omega_N > \dim G_v$, and therefore $s^* \omega_N \equiv 0$. Hence the assertions of the following Theorems 3 and 4 are trivial in the case $k + l > n$.

Theorem 4. *The inversion formula holds*

$$\int_\Gamma s^*\omega_N = c_\Gamma f(v), \tag{8}$$

where $s : G_v \to F_v$ is an arbitrary section; the integral is taken over an arbitrary cycle $\Gamma \subset G_v$ of dimension $N = (k-l)l$. The coefficient c_Γ depends only on the homology class to which the cycle Γ belongs.

We outline the proof of formula (8). In doing so we shall show that the derivation of formula (8) is easily reduced to the inversion formula for the ordinary Radon transform.

Let $v^0 = (v_1^0, \dots, v_l^0) \in E_{n,l}$. Choose an l-frame $b = (b^{(1)}, \dots, b^{(l)})$ in $(\mathbb{R}^n)'$ satisfying the following conditions: $(b^{(i)}, v_j^0) = \delta_j^i$, $i, j = 1, \dots, l$. Let $\mathbb{R}^{n-l} \subset \mathbb{R}^n$ be the annihilator of the set $\{b^{(1)}, \dots, b^{(l)}\}$. Define a function f_1 on

$$\underbrace{\mathbb{R}^{n-l} \times \dots \times \mathbb{R}^{n-l}}_{l}$$

by the formula

$$f_1(v_1', \dots, v_l') = f(v_1^0 + v_1', \dots, v_l^0 + v_l').$$

Consider the integrals of f_1 over the manifolds $h_1 \times \dots \times h_l$, where h_i is a $(k-l)$-dimensional plane in $\mathbb{R}^{n-l}$ $(1 \le i \le l)$:

$$\Phi(u_1', \alpha_{11}, \dots, \alpha_{1,k-l}; \dots; u_l', \alpha_{l,1}, \dots, \alpha_{l,k-l}) = \int f_1(u_1' + \alpha_{11} t_1^1 + \cdots$$

$$\cdots + \alpha_{1,k-l} t_1^{k-l}; \dots; u_l' + \alpha_{l,1} t_l^1 + \dots + \alpha_{l,k-l} t_l^{k-l}) \bigwedge_{j=1}^{l} \bigwedge_{i=1}^{k-l} dt_j^i,$$

u_i', $\alpha_{ij} \in \mathbb{R}^{n-l}$. From the function Φ we construct a differential form of degree N:

$$\tilde\omega = (\varkappa^{(1)} \wedge \dots \wedge \varkappa^{(l)}) \Phi(0, \alpha_{11}, \dots, \alpha_{1,k-l}; \dots; 0, \alpha_{l,1}, \dots, \alpha_{l,k-l}),$$

where

$$\varkappa^{(i)} = \bigwedge_{j=1}^{k-l} \left(\sum_{s=1}^{n-l} \frac{\partial}{\partial u_i^s} \, d\alpha_{ij}^s \right), \qquad 1 \le i \le l.$$

This differential form is defined on

$$\underbrace{G_{n-l,k-l} \times \cdots \times G_{n-l,k-l}}_{l}$$

and is closed. On the basis of the inversion formula for the Radon transform in $\mathbb{R}^{k-l+1}$ (1) we obtain that

$$\int_{\gamma\times\cdots\times\gamma} \tilde{\omega} = c_{\gamma\times\cdots\times\gamma} f_1(0)*,$$

where γ is the manifold of $(k-l)$-dimensional (oriented) subspaces contained in $\mathbb{R}^{k-l+1}$. On the other hand, if $\gamma_{i_1}, \dots, \gamma_{i_l}$ are arbitrary Schubert cells in $G_{n-l,k-l}$ such that $\dim \gamma_{i_1} + \cdots + \dim \gamma_{i_l} = N$ and

$$\gamma_{i_1} \times \cdots \times \gamma_{i_l} \neq \gamma \times \cdots \times \gamma,$$

then the restriction of $\tilde{\omega}$ to $\gamma_{i_1} \times \cdots \times \gamma_{i_l}$ is equal to zero (this follows directly from the definition of $\tilde{\omega}$). Hence we obtain

$$\int_{\Gamma} \tilde{\omega} = c_\Gamma f(v^0),$$

where

$$\Gamma \subset G_{n-l,k-l} \times \cdots \times G_{n-l,k-l}$$

is an arbitrary cycle of dimension N. It is easy to verify that on the submanifold

$$\operatorname{diag}\left(\underbrace{G_{n-l,k-l} \times \cdots \times G_{n-l,k-l}}_{l}\right)$$

the differential form $\tilde{\omega}$ coincides with $s^*\omega_N$ for a suitable choice of the section s. Therefore, if

$$\Gamma \subset \operatorname{diag}\left(G_{n-l,k-l} \times \cdots \times G_{n-l,k-l}\right),$$

then we obtain the required formula (8).

To compute the coefficient c_Γ in the inversion formula (8), we construct a cell decomposition of the manifold $G_v \simeq G_{n-l,k-l}$. Let $\widetilde{G}_{n-l,k-l}$ be the Grass–

* The coefficient $c_{\gamma\times\cdots\times\gamma}$ is equal to $(c_\gamma)^l$, where c_γ is the coefficient in the inversion formula

for the Radon transform in $\mathbb{R}^{k-l+1}$; see (1).

manifold of $(k-l)$-dimensional subspaces; let us give it its cell decomposition into Schubert cells ([4]). Our manifold $G_{n-l,k-l}$ of oriented subspaces is a two-sheeted covering over $\widetilde{G}_{n-l,k-l}$, and the inverse image of each Schubert cell splits into two connected components. The set of all these connected components is the desired cell decomposition of the manifold $G_v \cong G_{n-l,k-l}$. Take Schubert cells of dimension $N=(k-l)l$. Among them is the cell Γ_0, whose closure is the manifold of all $(k-l)$-dimensional subspaces in $\mathbb{R}^k$; the remaining cells of dimension N will be denoted by $\Gamma_1, \dots, \Gamma_s$; furthermore, denote by $\Gamma_i^{(1)}, \Gamma_i^{(2)}$ the connected components of the inverse image of the cell Γ_i in $G_{n-l,k-l}$ $(0 \le i \le s)$. It is readily established that: 1) the restrictions $s^*\omega_N$ to $\Gamma_1^{(\varepsilon)}, \dots, \Gamma_s^{(\varepsilon)}$ $(\varepsilon = 1, 2)$ are equal to zero; 2) the integrals of $s^*\omega_N$ over $\Gamma_0^{(1)}$ and $\Gamma_0^{(2)}$ coincide.

Theorem 5. Let $\Gamma \subset G_v$ be an arbitrary N-dimensional cycle. Then, if Γ is homologous to the chain $\sum(n_i'\Gamma_i^{(1)} + n_i''\Gamma_i^{(2)})$, where $n_i', n_i'' \in \mathbb{Z}$, then

$$c_\Gamma = c\,(n_0' + n_0''),$$

where $c \neq 0$ is a coefficient independent of Γ.

4. The results given above carry over without essential changes to the case of complex Grassmann manifolds. In this case, in the definition of the spaces $S_k(E_{n,l})$ and $S_l(E_{n,k})$, relations (1) and (2) must be replaced respectively by the following:

$$f(v\nu) = |\det \nu|^{-2k} f(v) \qquad \text{for every } \nu \in GL(l, \mathbb{C});$$

$$\varphi(u\mu) = |\det \mu|^{-2l} \varphi(u) \qquad \text{for every } \mu \in GL(k, \mathbb{C}).$$

The mapping $S_k(E_{n,l}) \to S_l(E_{n,k})$ is defined by the formula

$$\varphi(u) = \int_{G_{k,l}} f(ut)\,\sigma(t) \wedge \overline{\sigma(t)}.$$

Finally, the differential form $\omega_N = \varkappa_N \varphi$ should be replaced by

$$\omega_{N,N} = (\varkappa_N \wedge \overline{\varkappa_N})\varphi,$$

where $\overline{\varkappa_N}$ is obtained from $\varkappa_N$ by replacing all $\partial/\partial u_i^s$ and du_j^s respectively by $\partial/\partial \overline{u_i^s}$ and $d\overline{u_j^s}$.

Then all the theorems formulated by us for real manifolds remain valid in the complex case, and moreover without the additional assumption on the parity of $k-l$.

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Note: Figure translations are in progress. See original paper for figures.