ON SUPREMA OF FAMILIES OF \(H\)-CLOSED EXTENSIONS OF HAUSDORFF SPACES

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Abstract

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ON SUPREMA OF FAMILIES OF H-CLOSED EXTENSIONS OF HAUSDORFF SPACES

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1. In what follows, unless the contrary is stated, all spaces are assumed to be Hausdorff. By an H-closed extension of X we mean a triple $\langle X, i, Y\rangle$, where i is a homeomorphism of X onto an everywhere dense subset of Y, and Y is H-closed. Among such extensions we consider the usual quasi-order $\geqslant$, where $\langle X, i, Y\rangle \geqslant \langle X, i', Y'\rangle$ if there exists a continuous mapping $\gamma : Y \to Y'$, called admissible, such that $i' = \gamma \circ i$; then $\gamma(Y) = Y'$. This quasi-order in the usual way leads to a partial order on the equivalence classes of extensions of X, where two extensions fall into one class if and only if our γ is a homeomorphism. The family of these classes is a set, since the extensions are Hausdorff. We shall henceforth denote this set by Σ_X.

The main goal of the present paper is Theorem 1, which generalizes the analogous proposition of Engelking on bicompact extensions of Tikhonov spaces ([1], p. 127).

Theorem 1. *In the partially ordered set Σ_X of all equivalence classes of H-closed extensions of the space X, every nonempty subset has a supremum.*

2. Once and for all, from all the above-mentioned classes choose, for a representative, $\langle X, i_\lambda, i_\lambda X_\lambda\rangle$, where $i_\lambda(X) = X_\lambda$, and let $L = \{\lambda\}$ be the corresponding index set. Introduce the following notation for $S \subseteq L$:

$$P_S = \prod_{\lambda \in S} i_\lambda X_\lambda, \qquad \Delta_S = \{\{x_\lambda\} \mid i_\lambda(x) = x_\lambda,\ x \in X,\ \lambda \in S\},$$

denote the projection of P_S onto $i_\lambda X_\lambda$ by pr^S_λ, and its restriction $\mathrm{pr}^S_\lambda|_{\Delta_S}$ to Δ_S by p^S_λ. Then p^S_λ is a homeomorphism of Δ_S onto X_λ.

Lemma 1. *For $\lambda \in S \subseteq L$, the restriction pr^S_λ to the closure $(P_S)[\Delta_S]$ remains a mapping onto all of $i_\lambda X_\lambda$.*

Proof. It is necessary to show that for $y_{\lambda_0} \in i_{\lambda_0} X_{\lambda_0}$ there exists a point $y \in [\Delta_S]$ whose coordinate of index λ_0 is y_{λ_0}. Take the system of all neighborhoods of y_{λ_0} in $i_{\lambda_0} X_{\lambda_0}$, and let $\mathfrak{G}_{\lambda_0}$ be the system of their traces on X_{λ_0}. Let $\mathfrak{G}$ be the system of all preimages in X under the homeomorphism i_{λ_0} of the sets of the system $\mathfrak{G}_{\lambda_0}$. Then (together with $\mathfrak{G}_{\lambda_0}$) $\mathfrak{G}$ is a centered system of open sets of X, and extend $\mathfrak{G}$ in X to some maximal centered system of open sets $\mathfrak{G}^+$. Let $\mathfrak{G}^+_\lambda$ be the image of the system $\mathfrak{G}^+$ under the homeomorphism i_λ; then, in particular, $\mathfrak{G}^+_{\lambda_0} \supseteq \mathfrak{G}_{\lambda_0}$, whence it follows, in view of the centeredness of $\mathfrak{G}^+_{\lambda_0}$, that in the space $i_{\lambda_0} X_{\lambda_0}$ the point y_{λ_0} is a point of contact of every set of $\mathfrak{G}^+_{\lambda_0}$. Now let $\lambda \neq \lambda_0$, and for the centered system $\{O_\lambda(\Gamma) \mid \Gamma \in \mathfrak{G}^+_\lambda\}^*$, using the H-closedness of $i_\lambda X_\lambda$, choose a common point of contact y_λ. Then, in view of the equality $(i_\lambda X_\lambda)[O_\lambda(\Gamma)] = (i_\lambda X_\lambda)[\Gamma]$, y_λ is a common point of contact

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$*$ $O_\lambda(\Gamma)$ is the Shanin operator in $i_\lambda X_\lambda$: the largest open set in $i_\lambda X_\lambda$ containing Γ from X_λ.

sets in $\mathfrak{G}^+_\lambda$, also for $\lambda \neq \lambda_0$. Let us prove that $y = \{y_\lambda\} \in P_S$ (where y_{λ_0} is the initial point of $i_{\lambda_0} X_{\lambda_0}$) is a point of contact of Δ_S in P_S. Take a basic neighborhood of the point y:

$$O(H_{\lambda_1}, H_{\lambda_2}, \dots, H_{\lambda_n}), \qquad y_{\lambda_k} \in H_{\lambda_k}.$$

Then

$$H_{\lambda_k} \cap \Gamma \supset \Lambda \quad \text{for } \Gamma \in \mathfrak{G}_{\lambda_k},$$

whence

$$H_{\lambda_k} \cap \Gamma \cap X_{\lambda_k} \supset \Lambda, \quad \Gamma \in \mathfrak{G}_{\lambda_k}.$$

But then $X_{\lambda_k} \cap H_{\lambda_k}$ is open in X_{λ_k} and meets every $\Gamma \in \mathfrak{G}_{\lambda_k}$, and, in view of the maximality of $\mathfrak{G}^+_{\lambda_k}$ in X_{λ_k}, we have $X_{\lambda_k} \cap H_{\lambda_k} \in \mathfrak{G}^+_{\lambda_k}$. Let

$$G_k = i^{-1}_{\lambda_k}(X_{\lambda_k} \cap H_{\lambda_k});$$

then, for all k, $G_k \in \mathfrak{G}^+$, and

$$\bigcap_k G_k = G_0 \in \mathfrak{G}^+.$$

Take $x_0 \in G_0$ and the point $\{i_\lambda(x_0)\}$ in Δ_S. This point lies in $O(H_{\lambda_1}, H_{\lambda_2}, \dots, H_{\lambda_n}) \cap \Delta_S$, since

$$i_{\lambda_k}(x_0) \in i_{\lambda_k}(G_0) \subseteq i_{\lambda_k}(G_k) = H_{\lambda_k} \cap X_{\lambda_k} \subseteq H_{\lambda_k},$$

as was required to prove.

Lemma 2. For $S \subseteq L$, the closure Δ_S in P_S is an H-closed space.

Proof. First consider the special case $S = L$. Let $\Delta = \Delta_L$, $\Delta^+ = (P_L)[\Delta]$, and let $\widehat{\Delta^+}$ be some H-closed space in which Δ^+ is an everywhere dense subspace. Then, for the restriction

$$p_\lambda = \mathrm{pr}^L_\lambda \big|_\Delta,$$

the composition $p_\lambda^{-1} \circ i_\lambda$ is a homeomorphism of X onto Δ, independent of λ, which we denote by i. But then $\langle X, i, \widehat{\Delta}^+ \rangle$ is an H-closed extension of X, and take the H-closed extension equivalent to it from among the selected ones:

$$\langle X, i_{\lambda_1}, i_{\lambda_1} X_{\lambda_1} \rangle, \qquad \lambda_1 \in L,$$

which is possible, since we have chosen representatives from all equivalence classes! Then there exists a homeomorphism γ of the space $\widehat{\Delta}^+$ onto $i_{\lambda_1} X_{\lambda_1}$ such that $\gamma \circ i = i_{\lambda_1}$. But then

$$\gamma \circ p_{\lambda_1}^{-1} \circ i_{\lambda_1} = i_{\lambda_1},$$

and, multiplying both sides of this equality on the right by

$$i_{\lambda_1}^{-1} \circ p_{\lambda_1},$$

on the right-hand side we obtain

$$i_{\lambda_1} \circ i_{\lambda_1}^{-1} \circ p_{\lambda_1} = \mathrm{id}_{X_{\lambda_1}} \circ p_{\lambda_1} = p_{\lambda_1}^*,$$

while on the left-hand side we obtain

$$\gamma \circ p_{\lambda_1}^{-1} \circ i_{\lambda_1} \circ i_{\lambda_1}^{-1} \circ p_{\lambda_1} = \gamma \circ p_{\lambda_1}^{-1} \circ \mathrm{id}_X \circ p_{\lambda_1} = \gamma \circ p_{\lambda_1}^{-1} \circ p_{\lambda_1} = \gamma \circ \mathrm{id}_\Delta,$$

where $\gamma \circ \mathrm{id}_\Delta = \gamma|_\Delta$ is the restriction of γ to Δ. But p_{λ_1} is the restriction of $\mathrm{pr}_{\lambda_1}^L$ also to the set Δ, and since γ and $\mathrm{pr}_{\lambda_1}^L$ are defined and continuous on Δ^+, it follows that, coinciding on Δ, they also coincide on Δ^+; hence

$$\gamma(\Delta^+) = \mathrm{pr}_{\lambda_1}^L(\Delta^+).$$

But by Lemma 1

$$\mathrm{pr}_{\lambda_1}^L(\Delta^+) = i_{\lambda_1} X_{\lambda_1},$$

so that

$$\gamma(\Delta^+) = i_{\lambda_1} X_{\lambda_1}.$$

Since $\gamma(\Delta^+) = i_{\lambda_1} X_{\lambda_1}$, by the one-to-one character of γ we have

$$\widehat{\Delta}^+ \setminus |\Delta^+| = \Lambda \quad \text{and} \quad \Delta^+ = \widehat{\Delta}^+,$$

i.e. Δ^+ is H-closed, and our lemma is proved for the case $S = L$. Now let $S \subseteq L$; then P_S is a subproduct for P_L, and let pr_S be the projection of P_L onto P_S. Then

$$\mathrm{pr}_S(\Delta_L) = \Delta_S, \qquad \Delta_S \subseteq \mathrm{pr}_S\left((P_L)[\Delta_L]\right) \subseteq (P_S)[\mathrm{pr}_S \Delta_L] = (P_S)[\Delta_S],$$

and the H-closed space

$$\mathrm{pr}_S\left((P_L)[\Delta_L]\right),$$

containing Δ_S, lies in its closure in P_S and, consequently, coincides with it; whence it follows that $(P_S)[\Delta_S]$ is H-closed.

The **proof of the theorem** can now be obtained by proving that, for

$$\{\langle X, i_\lambda, i_\lambda X_\lambda\rangle \mid \lambda \in S \subseteq L\},$$

the required supremum is the class containing

$$\langle X, p_\lambda^{-1} \circ i_\lambda, (P_S)[\Delta_S]\rangle.$$

The proof of this fact is not difficult and, in the analogous situation (for bicompact extensions), is contained in Engelking ([1]); moreover, the analogue of our Lemma 2 required there is obtained at once, since the closure of a set in a bicompact space is bicompact!

3. Below, for an extension $\langle X, i, Y\rangle$ we shall assume $i = \mathrm{id}_X$, and we shall allow ourselves the inaccuracy of not distinguishing equivalent extensions both from one another and from the equivalence class containing them. From the method of proving the theorem there also follows the following.

Proposition. If, under the hypotheses of the theorem, the nonempty subset in Σ_X lies in the multiplicative class and in the class Ξ hereditary with respect to closed sets, then its supremum belongs to Ξ.

Lemma 3. If $\widetilde{X} = \sup \widetilde{X}$, $\widetilde{X} \subseteq \Sigma_X$, $\widetilde{X} = \{i_\lambda X \mid \lambda \in S\}$, then the family of admissible mappings

$$\gamma_\lambda : \widetilde{X} \to i_\lambda X, \qquad \lambda \in S,$$

separates the points of $\widetilde{X}$.

Proof. Let $y, y' \in \widetilde{X}$, $y \neq y'$. Without loss of generality we may assume

$$\widetilde{X} = (P_S)[\Delta_S] \subseteq \Pi\{i_\lambda X \mid \lambda \in S\},$$

so that

$$y = \{y_\lambda\}, \qquad y' = \{y'_\lambda\},$$

and

$$\gamma_\lambda = \mathrm{pr}_\lambda \big|_{[\Delta]}.$$

Then, since $y \neq y'$, there exists $\lambda_0 \in S$ such that

$$y_{\lambda_0} \neq y'_{\lambda_0},$$

whence

$$\gamma_{\lambda_0}(y) = \mathrm{pr}^S_{\lambda_0} \big|_{[\Delta_S]}(y) = y_{\lambda_0} \neq y'_{\lambda_0} = \gamma_{\lambda_0}(y').$$

* id_M is the mapping $x \mapsto x$, $x \in M$.

Corollary 1. *Among all H-closed extensions Σ_x of the space X there exists the greatest one, h_1X.*

An extension X^+ of the space X is called an extension of X **with normally situated remainder** (see, for example, ([2])), if in X^+ there is an open base the boundaries of whose members lie entirely in X.

Corollary 2. *The supremum in $\widetilde{\Sigma}_x$ of any nonempty family $\widetilde{\Sigma}$ of H-closed extensions of X with normally situated remainder has normally situated remainder. If X has bicompact extensions with normally situated remainder, then among them there is a greatest one, h_2X.*

Proof. Let $\widetilde{\Sigma} = \{i_\lambda X \mid \lambda \in S\}$ be our family, $\widetilde{X} = \sup \widetilde{\Sigma}$; let $\mathfrak{G}_\lambda$ be the family of all open sets in $i_\lambda X$ whose boundaries lie in X; let γ_λ be the admissible mapping of $\widetilde{X}$ onto $i_\lambda X$;

$$\mathfrak{G}^* = \{\gamma_\lambda^{-1}(G) \mid G \in \mathfrak{G}_\lambda,\ \lambda \in S\},$$

and, finally, let X^* be the space on the set $\widetilde{X}$ whose topology is given by the open base of all finite intersections of sets from $\mathfrak{G}^*$. Then X^* is a refinement of $\widetilde{X}$ and at the same time an extension of X. From Lemma 3 it follows easily that X^* is Hausdorff, and then $X^* \in \Sigma_x$, and for all $\lambda \in S$

$$i_\lambda X \leq X^* \leq \widetilde{X},$$

whence $X^* = \widetilde{X}$, and it remains only to prove that X^* has normally situated remainder. For this we verify that finite intersections of sets from $\mathfrak{G}^*$ have, in X^*, boundaries lying entirely in X. But since the boundary of an intersection is always contained in the union of the boundaries, it suffices to prove the assertion for sets from $\mathfrak{G}^*$. Thus, let $H = \gamma_\lambda^{-1}(G)$, $G \in \mathfrak{G}_\lambda$, and $x \in (X^* \setminus X) \cap CH$; then $\gamma_\lambda(x) = y \in G$ and $y \in i_\lambda X \setminus X$, since under an admissible mapping the remainder of an extension is carried into the remainder. In view of the definition of $\mathfrak{G}_\lambda$, the point y has in $i_\lambda X$ a neighborhood $U(y)$ such that $U(y) \cap G = \Lambda$, and we take $G' \subseteq U(y)$, $y \in G' \in \mathfrak{G}$. Then $\gamma_\lambda^{-1}(G') = H' \in \mathfrak{G}^*$, $x \in H'$, $H' \cap H = \Lambda$, as required. The second part of Corollary 2 now follows by an additional use of our proposition.

Corollary 3. *The supremum in Σ_x of any nonempty family of Urysohn H-closed extensions of X is a Urysohn H-closed extension. If X has Urysohn H-closed extensions, then among them there is a greatest one, h_3X.*

The proof follows from our proposition, since the class of Urysohn spaces is multiplicative and hereditary.

4. In view of ([3]), p. 60, our extension h_1X coincides with Katětov's extension τX, and Corollary 1 is already known. Since Fomin' s extension σX (see ibid.), according to Katětov' s theorem ([4]), can be characterized as an H-closed extension X^+ into which X is strictly embedded (the system of

sets of the form $O(\Gamma)$ forms a base of X^+) and which is at the same time hypercombinatorially (if Γ_1 and Γ_2 are open in X and $\Gamma_1 \cup \Gamma_2$ is everywhere dense in X, then $X^+ \setminus X \subseteq O(\Gamma_1) \cup O(\Gamma_2)$, for which it is necessary and sufficient that for every open set Γ in X one have $\mathrm{Fr}\, O(\Gamma) \subseteq X^*$), it is easy to see that σX is nothing other than the greatest H-closed extension of X with normally situated remainder and can be obtained from $h_1 X = \tau X$ as the "strict refinement" of $h_1 X$, defined on the set $h_1 X$ by the open base $\{O(\Gamma)\}$. As for the existence of bicompact extensions with normally situated remainder, according to the Freudental-Morita theorem (([2]), p. 364) for this it is necessary and sufficient that in X there exist a base with bicompact boundaries, so that our extension $h_2 X$ coincides

* The necessity of this, equivalent to hypercombinatoriality (formulated by us in dual form: in terms of open sets), was proved by Flachsmeyer (([2]), p. 363); the sufficiency is easy to verify.

with the Freudenthal compactification (see there). Finally, the second half of Corollary 3 for the case of a Tikhonov X was proved in ([5]), where $\rho X = h_3 X$. As for a non-Tikhonov X, a Urysohn H-closed extension may fail to exist even if X is regular, as is shown by examples of Hewitt ([6]) and Novak ([7]) of infinite T_3-spaces on which every real continuous function is constant (see also the construction of such spaces by Herrlich ([8])), since from the existence of the required extension it would follow, by Katětov' s theorem ([9]), that it can be compactified to a bicompactum. The advantage of our definitions of $h_1 X$, $h_2 X$, and $h_3 X$, however, is that in doing so we do not use any preliminary constructions of spaces. To be sure, apart from Theorem 1, for this we still need an existence theorem for an H-closed extension. But its simple proof, not using constructions, due to A. D. Aleksandrov ([10]), meets precisely this need. Along these lines one can also arrive very quickly, for a Tikhonov space, at its largest bicompact extension (Stone–Čech), especially if one defines a Tikhonov space as a set lying in a bicompactum (which at once ensures that it has at least one bicompact extension: its closure in the bicompactum).

In conclusion we note that, unfortunately, the authors of ([5,11,12]) were unaware of: the paper ([3]), where a characterization of τX in our sense is already given; Weinberg' s article ([13]), where for the first time an internal criterion for T_3-closedness of a T_2-space was given (coinciding with condition $R(i)$ from ([11])); and, finally, our paper ([14]), containing an existence theorem for noncompactifiable extensions of canonical (semiregular) T_2-spaces; the authors cited refer in this connection only to Banaschewski ([15]).

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Note: Figure translations are in progress. See original paper for figures.