BOUNDARY BEHAVIOR OF THE DERIVATIVE OF A UNIVALENT FUNCTION

MATHEMATICS

1970

SovietRxiv

Abstract

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UDC 517.54

MATHEMATICS

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BOUNDARY BEHAVIOR OF THE DERIVATIVE OF A UNIVALENT FUNCTION

(Presented by Academician L. S. Pontryagin, 3 VII 1970)

We study here the derivatives of functions $f(z)$ that are analytic and univalent in the unit disk $D: \ |z| < 1$. It is known, for example, see ([1]), that the derivative $f'(z)$ of a univalent function can have radial limits only on a set of measure zero on the circle $|z| = 1$. MacMillan observed that from one Koebe theorem (Verzerrungssatz) it follows directly that $f'(z)$ is a normal analytic function ([2]), so that, by MacLane' s theorem ([3]), $f'(z)$ has angular limits on an everywhere dense set on $|z| = 1$. A meromorphic function is also normal in D, but the dense set on $|z| = 1$ on which it has angular limits is countable. In Theorem 1 it is proved that the derivative $f'(z)$ of a univalent function $f(z)$ has radial limits (and, consequently, angular limits) on an uncountable subset of $|z| = 1$. Of interest is the class of univalent functions $f(z)$ whose derivatives $f'(z)$ have radial limits only on a set of measure zero on $|z| = 1$, and Theorems 2 and 3 are devoted to the study of these functions.

Lemma. *If $f(z)$ is analytic and univalent in $|z| < 1$, then $f'(z)$ is a normal analytic function in $|z| < 1$.*

Proof. From the proof of one Koebe theorem (Verzerrungssatz) it is known that

$$|f''(z)|/|f'(z)| \leq 6/(1 - |z|^2),$$

whence

$$|f''(z)|/[1 + |f'(z)|^2] \leq 6|f'(z)|/(1 - |z|^2)(1 + |f'(z)|^2) =$$

$$= 6/(1 - |z|^2)(|f'(z)|^{-1} + |f'(z)|)$$

or

$$|f''(z)|/[1 + |f'(z)|^2] \leq 3/(1 - |z|^2).$$

From this one may conclude (see ([2]), p. 87) that $f'(z)$ is normal in $|z| < 1$.

Theorem 1. *If $f(z)$ is analytic and univalent in $|z| < 1$, then the set of points on $|z| = 1$ at which $f'(z)$ has angular limits is uncountable.*

Proof. By the lemma, $f'(z)$ is normal in $|z| < 1$ and has angular limits on a dense subset of $|z| = 1$. Further, one may restrict oneself to the case where $f'(z)$ has radial limits on a dense set of measure zero on $|z| = 1$. Since $f'(z)$ is different from 0 and ∞ in $|z| < 1$, for every positive integer n each component of the open set $H_n = \{z : \ |f'(z)| < 1/n\}$ is simply connected. Further, the intersection M of the set ∂G_n—the boundary of G_n—with $|z| = 1$ is nonempty. We have $M = M_a \cup M_i$, where M_a is the set of accessible points from G_n; M, M_i are the sets of inaccessible points from G_n. If $z = g(t)$ is an arbitrary conformal mapping of the domain $|t| < 1$ onto G_n, then, as is known, radial limits of $g(t)$ exist almost everywhere on $|t| = 1$, so that almost all radii of the disk $|t| < 1$ are mapped onto accessible boundary points of the set

G_n. In other words, since the unattainable boundary points of the set G_n correspond to a set of zero measure on $|z| = 1$, the set M_i has zero harmonic measure with respect to G_n. Further, the obvious generalization of Fatou' s theorem to the domain G_n shows that the linear measure M_a is equal to zero, and from Carleman' s principle it follows that M_u has zero harmonic measure with respect to G_n, whence the harmonic measure of the set $M = M_a \cup M_i$ with respect to G_n is also equal to zero.

We now assert that G_n contains at least one component H_{n+1}. If this is not so, then $1/(n+1) \le |f'(z)| < 1/n$ everywhere in G_n, and since $|f'(z)| < 1/n$ at all boundary points ∂G_n, except for some set of zero harmonic measure with respect to G_n, $f'(z)$ reduces in G_n to a constant of modulus $1/n$, which is impossible. Consequently, for every n each component H_n contains at least one component H_{n+1}.

Therefore, starting with a given component G_n, we can form a nested system of components $G_n \supset G_{n+1} \supset ...$ and construct a polygonal path L with an "end" on $|z| = 1$, along which $f'(z) \to 0$. The "end" E of the path L reduces to a single point, for otherwise one could indicate a nondegenerate arc E' of E together with a sequence of pairwise nonintersecting subarcs of L converging to E', in such a way that $|f'(z)| < \varepsilon_n$ on L_n and $\lim \varepsilon_n = 0$. The sequence $\{L_n\}$ is a sequence of Koebe for the normal function $f'(z)$, so that $f'(z) \equiv 0$. Consequently, E reduces on $|z| = 1$ to a single point p, and 0 is a possible asymptotic value of $f'(z)$ at p.

Let now G_n be an arbitrary component of the open set H_n. Let $z = g(t)$ be a conformal mapping of the disk $|t| < 1$ onto the simply connected domain G_n, and let M be the intersection of ∂G_n with $|z| = 1$. Since M has harmonic measure zero with respect to G_n and since $|f'(z)| = 1/n$ at all points of ∂G_n lying in $|z| < 1$, the function $F(t) = nf'(g(t))$ is analytic and bounded, $|F(t)| < 1$ for $|t| < 1$, and it has radial limits of modulus 1 almost everywhere on $|t| = 1$. Since $F(t)$ does not vanish in $|t| < 1$, $F(t)$ can be represented in the form

$$F(t) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{is}+t}{e^{is}-t}\,d\mu(s) + i\alpha\right\}, \tag{1}$$

where $\mu(s)$ is a monotone nonincreasing function with $\mu'(s) = 0$ almost everywhere and where α is a real constant. Suppose that the set of singularities of $\mu(s)$ is not perfect; then $\mu(s)$ has at least one isolated singularity, which may be set equal to $s = 0$. Then along the radius from $|t| < 1$ leading to the point $t = 1$, we have

$$|F(t)| = O\left\{\exp\frac{|t|+1}{|t|-1}\right\}. \tag{2}$$

Since this radius is transformed under the mapping $z = g(t)$ into a path from G_n ending at the point P on $|z| = 1$, along which $f'(z)$ tends to zero, it follows from the normality of $f'(z)$ that $f'(z)$ tends to zero uniformly in some angle at the point P. From $F(t) = nf'(z)g'(t)$ and from Koebe' s theorem (Verzerrungssatz) it follows that $|f'(z)g'(t)|$ cannot tend to zero with the estimate (2), so that $\mu(s)$ in (1) cannot have an isolated singularity. Since a nonconstant singular function has infinite derivative on an uncountable set, we have thereby proved that $\mu'(s) = -\infty$ on an uncountable set. Along the radius leading to any point of this set, $F(t)$ tends to zero, and the image of each of these radii under the mapping $z = g(t)$ is a path in G_n with endpoint at a point of $|z| = 1$, along which $f'(z)$ tends to zero. Since $f'(z)$ tends to zero uniformly in an angle at an arbitrary point of

$|z| = 1$, at which the path along which $f'(z)$ tends to zero terminates, then it is clear that two radii along which $F(t)$ tends to zero cannot, under the mapping $z = g(t)$, be mapped onto two paths ending at the same point on $|z| = 1$. Consequently, the set of points of $|z| = 1$ at which zero can be an angular limiting value of the function $f'(z)$ is countable, and the theorem is proved.

Remark. From an almost identical argument it follows that the set of points on $|z| = 1$ at which ∞ can be an angular limiting value of the function $f'(z)$ is also countable.

Theorem 2. *Let $f(z)$ be analytic and univalent in $|z| < 1$, and let $f'(z)$ have radial limiting values only on a set of measure zero on $|z| = 1$. Then every point on $|z| = 1$ is a point of condensation of the set of points of $|z| = 1$ at which 0 can be a radial limiting value of the function $f'(z)$.*

Remark. In Theorem 2 the value 0 may be replaced by the value ∞.

It is natural to ask whether there exist radial limiting values different from 0 and ∞. In Theorem 3 we give a sufficient condition for the existence of other radial limiting values. We shall say that the points $\{z_k(\alpha)\}$ from $|z| < 1$ are α-values of $f'(z)$ if $f'(z_k(\alpha)) = \alpha$. A sequence of α-values is said to satisfy the Blaschke condition if

$$\sum(1-|z_k(\alpha)|)<\infty. \tag{3}$$

It is known that if the set of values α for which (3) holds has positive logarithmic capacity, then $f'(z)$ has bounded characteristic in $|z|<1$; consequently, $f'(z)$ has radial limits almost everywhere on $|z|=1$. Hence, if one assumes that $f'(z)$ has radial limits only on a set of measure zero, then $f'(z)$ cannot have bounded characteristic, and therefore the set of values α for which the α-values $z_k(\alpha)$ satisfy (3) must be of zero capacity.

Theorem 3. *Let $f(z)$ be analytic and univalent in $|z|<1$, and let $f'(z)$ have radial limits only on a set of measure zero on $|z|=1$. If the α-values $\{z_k(\alpha)\}$ of the function $f'(z)$ satisfy (3), then α is an angular limiting value of $f'(z)$ on every arc of the circle $|z|=1$.*

As stated, MacMillan pointed out the lemma to us. Ch. Pommerenke informed us that he knows another proof of Theorem 1. It is unknown whether, in Theorem 1, the set of points at which $f'(z)$ has radial limits has positive capacity.

Received
3 VI 1970

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Note: Figure translations are in progress. See original paper for figures.