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Abstract

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MATHEMATICS

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ON AN INTERNAL CHARACTERISTIC OF RELATIVELY OPEN CONVEX SETS

(Presented by Academician L. V. Kantorovich on 26 I 1970)

The article establishes a connection between relatively open convex sets in real vector spaces* and abstract axial structures, consideration of which is led to by the formalization of the elementary-geometric concept of an axis.

Let us agree to say that an **axial structure** is defined on a set Q if in it there is singled out some collection R of nonempty ordered subsets (**axes**) and, moreover, the following conditions (axioms) are satisfied:

1. Each axis $\vec{\Pi} \in R$ is a complete (linearly) ordered continuous set (all its sections are Dedekind) without a least and without a greatest element.
2. Whatever two (distinct) elements–points $x, y \in Q$ may be, in R there exists one and only one axis $\vec{\Pi} = \Pi(x, y)$ such that $x \in \Pi$, $y \in \Pi$, $x < y$, where $<$ denotes the precedence relation generated by the order relation $\leqslant$ on the axis $\vec{\Pi}$.
3. The axes $\vec{\Pi}(x, y)$ and $\Pi(y, x)$, corresponding to any two (distinct) points $x, y \in Q$, consist of the same elements.
4. If three points $x, y, z \in Q$ do not lie on one axis, then for any $u \in \vec{\Pi}(x, y)$, $v \in \vec{\Pi}(y, z)$ such that on the corresponding axes $y < u$, $y < v < z$, on the axis $\Pi(u, v)$ there will be found a point w, also belonging to the axis $\vec{\Pi}(x, z)$, and satisfying on it the relation $x \leqslant w \leqslant z$.

Sets with axial structures (with a somewhat different, but equivalent, system of axioms), under various names (**geometry without dimension**, **continuous models of betweenness**, **continuous geometry of order**), were considered already by Pasch (1882), Peano (1889), Hilbert (1899), Veblen (1904), and others (see ([1]), pp. 253-275). Among recent publications we note ([2–4]).

In each set Q with an axial structure R, the notions of a **line**, a **segment**, an **affine manifold**, a **convex set**, the **affine** and **convex hull** of an arbitrary set $A \subset Q$, as well as the notions of a **relatively open** and **(*)-open **set, are defined in the obvious manner. Note that the axial structure R, defined in Q, obviously induces an axial structure R' in every**

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relatively open convex set $Q' \subset Q$. In this case the set Q with axial structure R is called an extension** of the set Q' with axial structure R'.

* A set M in a real vector space E is called open relative to its affine hull L, or, more briefly, **relatively open**, if for any line $\Pi \subset L$ the set $\Pi \cap L$ is open on Π. If, moreover, $L = E$ or $L = \emptyset$, then the set M is called **(*)-open**.

Further, points of a set $A \subset Q$ are said to be **affinely independent** if none of these points is contained in the affine hull (aff. hull) of the others. If the points of a set A are affinely independent and $x_0 \in \operatorname{aff.hull} A$, then the points of the set $A' = A \cup \{x_0\}$ also turn out to be affinely independent. Owing to this, among the affinely independent subsets A of an arbitrary set $M \subset Q$ there are maximal ones. Each such subset is called an **affine basis** of the set M. It can be shown that all affine bases of the set M have the same cardinality. The sets that supplement affine bases of the set M to affine bases of all of Q also have the same cardinality. This makes it possible to introduce the notions of **affine dimension** (dim) and **affine codimension** (codim) for an arbitrary set $M \subset Q$. In order not to contradict customary usage, $\dim M$ is taken to be the cardinality of an affine basis of the set M with one element excluded, while $\operatorname{codim} M$ is taken to be the cardinality of the complement of an affine basis of the set M to an affine basis of all of Q. Below, two-dimensional affine manifolds, called **planes**, and affine manifolds of codimension one, called **hyperplanes**, are singled out. We note that if H is a hyperplane in Q, then $Q \setminus H$ is uniquely representable as the union of two disjoint $(*)$-open convex sets Q_1 and Q_2, called **open half-spaces** corresponding to the hyperplane H. Moreover, the segment joining any two points $x_1 \in Q_1$ and $x_2 \in Q_2$ has a common point with H.

In every real vector space an axial structure is defined in a natural way. As axes, evidently, it suffices to take the nondegenerate straight lines ordered by increasing values of the parameter t,

$$\{z = x + t(y - x) : t \in (-\infty, +\infty)\}.$$

In this connection, in ([4]) abstract sets with fixed axial structures are called **generalized vector spaces** (g.v.s.).

From what was said above it is clear that the natural axial structure in a real vector space E induces an axial structure in every relatively open convex set $M \subset E$. Thus, relatively open convex sets of a real vector space with their natural axial structures constitute one of the possible realizations of g.v.s. We shall show that the indicated realization is, in a certain sense, exhaustive.

First of all, let us single out the class of g.v.s. isomorphic to axial structures of real vector spaces. For this purpose we introduce in a g.v.s. the concept of parallelism of axes. We shall agree to say that an axis $\vec{\Pi}_0$ is **parallel** to an axis $\vec{\Pi}$, and write $\vec{\Pi}_0 \parallel \vec{\Pi}$, if $\vec{\Pi}_0 \cap \vec{\Pi} = \emptyset$ and, for any

$$x \in \vec{\Pi}_0, \quad y \in \vec{\Pi}_0, \quad x < y, \quad z \in \vec{\Pi}, \quad u \in \vec{\Pi}(y, z), \quad y < u < z,$$

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the axes $\vec{\Pi}(x,u)$ and $\vec{\Pi}$ have a common point v, which on these axes satisfies the relations $u < v,\ z < v$.

It can be shown that, whatever the g.v.s. (Q,R), an axis $\vec{\Pi} \in R$, and a point $x \in \vec{\Pi}$, in R there is one and only one axis $\vec{\Pi}_0$ satisfying the conditions $x \in \vec{\Pi}_0,\ \vec{\Pi}_0 \parallel \vec{\Pi}$. The g.v.s. under consideration will be called **Euclidean** (e.g.v.s.) if, in addition to conditions 1–4, the following axiom is also satisfied:

5. If $\vec{\Pi}(x,y) \parallel \vec{\Pi}(z,u)$, then $\vec{\Pi}(y,x) \parallel \vec{\Pi}(u,z)$.

Real vector spaces with their natural axial structures are, evidently, e.g.v.s. By means of constructions analogous to those considered in another connection in ([5]) (pp. 112–122), it is not difficult to verify that the following converse assertion is also true.

If an e.g.v.s. (Q,R) is more than two-dimensional (contains four affinely independent points), then, fixing an arbitrary zero point

$0 \in Q$, one can (and moreover uniquely) introduce in Q the structure of a real vector space in such a way that the natural axial structure of this vector space coincides with the original axial structure in Q.

This means that if the abstract a.v.s. (Q,R) under consideration is of dimension greater than two, then the question of its realization as a relatively open convex set of a real vector space reduces to checking the possibility of extending the a.v.s. (Q,R) to an e.o.v.s.

As for two-dimensional a.v.s., the possibility of extending them to an e.o.v.s. still does not imply the existence of the desired realization. Moreover, there exist two-dimensional a.v.s. in which the relation of parallelism of axes is not symmetric, and such a.v.s. obviously cannot be extended to an e.o.v.s. The corresponding example is considered in ([4]) (p. 1124).

Let us now note that in a.v.s. of dimension greater than two which admit an extension to an e.o.v.s., as is not difficult to verify, the following proposition from ([2]) (pp. 85—89) holds.

Lemma on roofs. *If the lines Π_1 and Π_2 do not coincide and lie in one plane P, then for any points $x_1, x_2 \in P$ the lines*

$$\Pi_{x_1} = \bigcap_{i=1,2} \operatorname{aff.env.}(\Pi_i \cup \{x_1\}), \qquad \Pi_{x_2} = \bigcap_{i=1,2} \operatorname{aff.env.}(\Pi_i \cup \{x_2\})$$

also lie in one plane.

The principal result of the present paper is the following

Theorem. *If in an a.v.s. (Q,R) of dimension greater than two the lemma on roofs holds, then it admits an extension to an e.o.v.s.*

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Let us indicate the main milestones of the proof. Fix in Q some hyperplane H, the open half-spaces Q_1 and Q_2 corresponding to it, and three affinely independent points $x_1, x_2, x_3 \in H$. By S_i $(i = 1, 2, 3)$ denote the pencil of lines intersecting the hyperplane H at the point x_i (passing through this point and having no other common points with H), and consider the set $\widetilde{Q}$ of triples $\tilde{x} = (\Pi_1, \Pi_2, \Pi_3)$ of lines $\Pi_1 \in S_1$, $\Pi_2 \in S_2$, $\Pi_3 \in S_3$ such that every two of them lie in one plane. It is clear that each element $\tilde{x} \in \widetilde{Q}$ is uniquely determined by specifying any two lines of the corresponding triple. Moreover, if some two lines of the triple $(\Pi_1, \Pi_2, \Pi_3) \in \widetilde{Q}$ have a common point, then the third line also passes through this point. Assigning to points $x \in Q \setminus H$ the elements $f(x) \in \widetilde{Q}$ determined by the lines $\Pi(x_1, x)$, $\Pi(x_2, x)$, $\Pi(x_3, x)$, we obtain a one-to-one (injective) mapping of the set $Q \setminus H = Q_1 \cup Q_2$ into $\widetilde{Q}$.

Relying on the lemma on roofs, in $\widetilde{Q}$ one can define a collection of axes $\widetilde{R}$ in such a way that conditions 1–5 are satisfied. The e.o.v.s. $(\widetilde{Q}, \widetilde{R})$ thereby obtained turns out to be related to the original a.v.s. (Q, R) in the following way. If a line $\Pi \subset Q$ has no common points with the hyperplane H, then its image $f(\Pi) \subset \widetilde{Q}$ coincides with the intersection of the set $f(Q \setminus H)$ with some line $\widetilde{\Pi} \subset \widetilde{Q}$ and is an open segment, an open ray (without its vertex), or the whole line $\widetilde{\Pi}$. If, however, the line $\Pi \subset Q$ intersects the hyperplane H at some point x_0, then the set $f(\Pi \setminus \{x_0\})$ coincides with the intersection of the set $f(Q \setminus H)$ with some line $\widetilde{\Pi} \subset \widetilde{Q}$ and is the union of two open rays of the line $\widetilde{\Pi}$ without common points. Moreover, if the lines Π_1 and Π_2 intersect the hyperplane H at the points x_0' and x_0'', then the sets $f(\Pi_1 \setminus \{x_0'\})$ and $f(\Pi_2 \setminus \{x_0''\})$ lie on parallel lines in $\widetilde{Q}$ if and only if $x_0' = x_0''$. Further, if $x, y, u, v \in Q \setminus H$, with the points u, v lying on the axis $\Pi(x, y)$ and on it $u < v$, then the images of these points on the axis $\widetilde{\Pi}(f(x), f(y))$ satisfy the relation $f(u) < f(v)$ if the segment uv does not intersect H, and the relation $f(v) < f(u)$ in the opposite case. Finally, the affine hulls of the sets $f(Q_1)$ and $f(Q_2)$ coincide with all of $\widetilde{Q}$.

From the facts presented it follows, in particular, that the sets $f(Q_1)$ and $f(Q_2)$, which have no common points in the a.o.v.p. $(\widetilde{Q}, \widetilde{R})$, are convex and $(*)$-open. Consequently, there exists a hyperplane $\widetilde{H} \subset \widetilde{Q}$ such that the corresponding open half-spaces $\widetilde{Q}_1$ and $\widetilde{Q}_2$ contain the sets $f(Q_1)$ and $f(Q_2)$, respectively. Fix in $\widetilde{H}$ some point and denote by $\widetilde{H}'$ the pencil of lines intersecting the hyperplane $\widetilde{H}$ at this point. For each point $x \in H$, on the basis of what was said above, in $\widetilde{H}'$ there is one and only one line parallel to the f-images of open rays with vertex at the point x.

Consider the set $\widetilde{Q}' = (\widetilde{Q} \setminus \widetilde{H}) \cup \widetilde{H}'$, and extend the injective mapping f of the set $Q \setminus H$ into $\widetilde{Q} \setminus \widetilde{H}$ to an injective mapping of the set Q into $\widetilde{Q}$. For this, to each point $x \in H$ we assign the uniquely determined element $f(x) \in \widetilde{H}'$. With the aid of the axial structure $\widetilde{R}$ in $\widetilde{Q}$, define a collection of axes $\widetilde{R}'$ in $\widetilde{Q}'$. To this end, include in $\widetilde{R}'$, without change, the axes $\widetilde{\Pi} \in \widetilde{R}$ that have no common

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points with the hyperplane $\widetilde{H}$. Each axis $\widetilde{\Pi} \in \widetilde{R}$ that intersects the hyperplane $\widetilde{H}$ at a certain point $\tilde{x}_0$ is replaced by the axis $\widetilde{\Pi}' = (\widetilde{\Pi} \setminus \{\tilde{x}_0\}) \cup \{\tilde{x}_0'\}$, where as $\tilde{x}_0'$ one chooses the line of the pencil $\widetilde{H}'$ parallel to the axis $\widetilde{\Pi}$. The order relation on the new axis $\widetilde{\Pi}'$ is established so that, if on the axis $\widetilde{\Pi}$

$$\tilde{y}_1 < \tilde{y}_2 < \tilde{x}_0 < \tilde{z}_1 < \tilde{z}_2,$$

then on the axis $\widetilde{\Pi}'$ the relation

$$\tilde{z}_1 < \tilde{z}_2 < \tilde{x}_0' < \tilde{y}_1 < \tilde{y}_2$$

holds. Finally, instead of the axes $\widetilde{\Pi} \in \widetilde{R}$ situated in $\widetilde{H}$, the axes $\widetilde{\Pi}'$, which are naturally ordered two-dimensional subpencils of the pencil $\widetilde{H}'$, are included in $\widetilde{R}'$.

It is not difficult to verify that the set $\widetilde{Q}'$, with the collection of axes $\widetilde{R}'$ introduced in the indicated way, satisfies conditions 1-5. At the same time the set $f(Q) \subset \widetilde{Q}'$ is convex and relatively open (moreover, $(*)$-open), and the axial structure $\widetilde{R}''$ induced in it turns it into an o.v.p. $(f(Q), \widetilde{R}'')$, isomorphic to the original o.v.p. (Q, R). This means that, if the points $x \in Q$ are identified with their images $f(x) \in \widetilde{Q}'$, then the constructed a.o.v.p. $(\widetilde{Q}', \widetilde{R}')$ may be taken as the desired extension of the o.v.p. (Q, R), and the theorem is proved.

To complete the investigation of the question posed, concerning the relation between abstract o.v.p.' s, on the one hand, and relatively open convex sets in real vector spaces, on the other, it remains to determine whether the lemma on roofs is valid in any o.v.p. of dimension greater than two. For o.v.p.' s of dimension greater than three, the validity of the indicated lemma is verified with the aid of the simplest relations between the dimensions of certain affine manifolds. For three-dimensional o.v.p.'s this lemma is considered in ([2]) (pp. 85-89). However, there are certain gaps in its proof which we have not been able to fill. Therefore, exercising the necessary caution, we shall consider that for the three-dimensional case the question still remains open.

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Note: Figure translations are in progress. See original paper for figures.

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