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Abstract

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MATHEMATICS

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SOME PATHOLOGICAL MAXIMAL IDEALS IN THE ALGEBRA OF MEASURES ON A COMPACT GROUP

(Presented by Academician P. S. Aleksandrov on 26 I 1970)

§ 1. In this note we consider the algebra $\mathbf{M}(G)$ of all regular complex Borel measures on a compact group G. With respect to the operations of addition, scalar multiplication, multiplication (convolution), given by the formula

$$\mu * \nu(E) = \int_G \mu(Et^{-1})\, d\nu(t),$$

and the norm $\|\mu\| = |\mu|(E)$, $\mathbf{M}(G)$ is a Banach algebra. It has as identity the measure ε_e of mass 1 concentrated at the identity e of the group G, and is commutative if and only if the group G is commutative. The mapping $\mu \mapsto \mu^{\sim}$, where $\mu^{\sim}(E) = \mu(E^{-1})$, is an involution in $\mathbf{M}(G)$ (the elementary theory of the algebra $\mathbf{M}(G)$ is developed in ([1])). The algebra $\mathbf{M}(G)$ contains the algebra of measures absolutely continuous with respect to Haar measure λ as a closed two-sided ideal. This algebra can be identified with the algebra $\mathfrak{L}_1(G)$ of all λ-integrable functions on G. For the algebra $\mathfrak{L}_1(G)$ it is well known (see, for example, ([1]), § 38) that all its closed ideals, both one- and two-sided, can be described in terms of representations of the group G. In particular, every maximal two-sided proper ideal in $\mathfrak{L}_1(G)$ has the form

$$\left\{ f \in \mathfrak{L}_1(G) : \int_G U_x f(x)\, d\lambda(x) = 0 \right\} = \mathfrak{I}_U,$$

where $x \mapsto U_x$ is some continuous irreducible unitary representation of the group G. The quotient algebra $\mathfrak{L}_1(G)/\mathfrak{I}_U$ is isomorphic to the algebra of all linear operators on the representation space of U, and therefore is finite-dimensional.

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For several years the question remained open whether the algebra $\mathbf{M}(G)$ also has the latter property: is it true that a simple homomorphic image of $\mathbf{M}(G)$ is necessarily finite-dimensional? In this note a negative answer is given to this question: there exist compact groups G for which $\mathbf{M}(G)$ contains such a maximal two-sided ideal J that $\mathbf{M}(G)/J$ is not only infinite-dimensional, but also contains a "scattered" set of any prescribed cardinality.

§ 2. We begin the proof by constructing maximal ideals of some algebras of operators. Let H be a finite-dimensional Hilbert space of dimension d, let $\mathfrak{B}(H)$ be the algebra of all linear operators on H, and let $\mathfrak{U}(H)$ be the compact group of unitary operators in $\mathfrak{B}(H)$. For any $A \in \mathfrak{B}(H)$ we shall write AA^* in the form $\sum_{k=1}^{n} a_k P_k$, where P_k are the projections onto pairwise orthogonal one-dimensional subspaces of H, and a_k are–

nonnegative real numbers. For $1 \le p < \infty$ let

$$\|A\|_{\Phi_p} = \left(\sum_{k=1}^{d} a_k^{p/2}\right)^{1/p}$$

and

$$\|A\|_{\Phi_\infty} = \max\{a_1^{1/2}, a_2^{1/2}, \dots, a_d^{1/2}\}.$$

von Neumann [2] showed that all the functions $\|\cdot\|_{\Phi_p}$ are norms on $\mathfrak{B}(H)$. It is easy to show that $\|\cdot\|_{\Phi_\infty}$ is also the operator norm on $\mathfrak{B}(H)$.

Let now I be a nonempty set of indices, and for each $i \in I$ let H_i be a finite-dimensional Hilbert space of dimension d_i. Finally, let $\mathfrak{C}_\infty(I)$ be the set of all elements $A = (A_i)_{i\in I}$ of the Cartesian product $\prod_{i\in I} \mathfrak{B}(H_i)$ such that

$$\|A\| = \sup_{i\in I} \|A_i\|_{\Phi_\infty}.$$

With respect to coordinatewise operations and the norm defined above, $\mathfrak{C}_\infty(I)$ is, obviously, a C^*-algebra with identity, commutative if and only if all $d_i = 1$.

§ 3. Following Wright [3], we construct maximal two-sided ideals in $\mathfrak{C}_\infty(I)$.

3.1. **Theorem 1.** *Let $\mathcal{U}$ be any ultrafilter on the set I, and let $\mathfrak{I}_U$ be the set of all such $A \in \mathfrak{C}_\infty(I)$ that*

$$\lim_U \frac{1}{d_i}\|A\|_{\Phi_1} = 0.$$

Then $\mathfrak{I}_U$ is a maximal two-sided ideal in $\mathfrak{C}_\infty(I)$, and, conversely, every maximal two-sided ideal in $\mathfrak{C}_\infty(I)$ has this form for a suitable choice of an ultrafilter $\mathcal{U}$ on the set of indices I.

3.2. **Theorem 2.** *Suppose that for some positive integer m the set $\{i \in I : d_i \le m\}$ belongs to the ultrafilter $\mathcal{U}$. Then the algebra $\mathfrak{C}_\infty(I)/\mathfrak{I}_U$ is isomorphic to the algebra $\mathfrak{B}(L)$ for some Hilbert space L of dimension m.*

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Let X be a metric space and let ρ be its metric. A subset $S \subset X$ is called scattered if there exists a positive number α such that $\rho(x, y) \geq \alpha$ for any pair x, y of distinct elements of S.

3.3. **Theorem 3.** *Suppose that for every positive integer m the set $\{i \in I : d_i > m\}$ belongs to $\mathcal{U}$. Then the quotient algebra $\mathfrak{C}_\infty(I)/\mathfrak{I}_U$ contains a scattered set S of cardinality $2^{\aleph_0}$ such that every element of S has norm 1. In particular, the quotient algebra is infinite-dimensional.*

3.4. **Theorem 4.** *Let $\mathfrak{m}$ be an infinite cardinal number. There exist sets of indices I of cardinality $\mathfrak{m}$, families of finite-dimensional Hilbert spaces $\{H_i\}_{i \in I}$, and an ultrafilter $\mathcal{V}$ on the set I, such that the quotient algebra $\mathfrak{C}_\infty(I)/\mathfrak{I}_V$ contains a scattered set of cardinality $2^{\mathfrak{m}}$, all elements of which have norm 1.*

§ 4. **Applications to the algebra** $\mathrm{M}(G)$**.** Let again G be a compact group. The object dual to the group G is the set Σ of equivalence classes of continuous unitary irreducible representations of the group G. Thus, each $\sigma \in \Sigma$ is a class of pairwise equivalent d_σ-dimensional continuous unitary irreducible representations of G, where d_σ is some positive integer. In each class $\sigma \in \Sigma$ choose a representative $U^{(\sigma)} \in \sigma$. The representation $U^{(\sigma)}$ acts on the d_σ-dimensional Hilbert space H_σ. In H_σ choose an orthonormal basis $\xi_1, \xi_2, \dots, \xi_{d_\sigma}$. Let D_σ be the conjugate-linear operator on H_σ such that

$$D_\sigma \left(\sum_{k=1}^{\sigma} a_k \xi_k \right) = \sum_{k=1}^{\sigma} \overline{a_k} \xi_k.$$

For $\mu \in \mathrm{M}(G)$ we define the Fourier–Stieltjes transform $\hat{\mu}$ of the element μ as the element of $\mathfrak{C}_\infty(\Sigma)$ such that

$$\langle \hat{\mu}(\sigma)\xi, \eta \rangle = \int_G \langle D_\sigma U_x^{(\sigma)} D_\sigma \xi, \eta \rangle \, d\mu(x)$$

for all $\xi, \eta \in H_\sigma$ and all $\sigma \in \Sigma$. The mapping $\mu \mapsto \hat{\mu}$ is a norm-nonincreasing involutive isomorphism of $\mathrm{M}(G)$ into $\mathfrak{C}_\infty(\Sigma)$. For any nonempty

subsets $P \subset \Sigma$ the mapping $\mu \mapsto (\mu(\sigma))_{\sigma \in P} = \hat{\mu}|P$ is a norm-nonincreasing involutive homomorphism of $M(G)$ into $\mathfrak{C}_\infty(G)$.

4.1. Definition. A nonempty subset $P \subset \Sigma$ is called a **Sidon set** if the mapping $\mu \mapsto \hat{\mu}|P$ is a homomorphism of $M(G)$ onto $\mathfrak{C}_\infty(P)$.

4.2. Example. For any indexed family $\{H_i\}_{i \in I}$ of finite-dimensional Hilbert spaces, the product

$$L = \prod_{i \in I} U(H_i)$$

is a compact group, and the mapping $(U_i)_{i \in I} \mapsto U_{i_0}$ is a d_{i_0}-dimensional continuous unitary irreducible representation of L for any fixed $i_0 \in I$. In (1) it was proved (37.5) that the set of all such representations defines a Sidon set in the dual object of L.

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4.3. Theorem 5. Let G be a compact group and let P be a Sidon set in Σ such that

$$\sup\{d_\sigma : \sigma \in P\} = \infty.$$

Then $M(G)$ contains a maximal two-sided ideal $\mathfrak{I}_u$, the quotient algebra $M(G)/\mathfrak{I}_u$ by which contains a scattered set as in (3.3).

4.4. Theorem 6. Let L be the group defined in (4.2), and suppose that

$$\sup\{d_i : i \in I\} = \infty.$$

Then $M(L)$ contains a maximal two-sided ideal with the properties described in (4.3).

4.5. Theorem 6. Let $\mathfrak{m}$ be an infinite cardinal number. There exists a product L, as in (4.2), and such a maximal two-sided ideal $\mathfrak{I}$ in $M(L)$ that $M(L)/\mathfrak{I}$ contains a scattered set P as in (3.4).

Theorems 5 and 6 follow directly, respectively, from Theorems 3 and 4. It is only necessary to note that the Fourier-Stieltjes transform, restricted to P, is a homomorphism "onto."

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REFERENCES

1. E. **Hewitt**, K. A. **Ross**, *Abstract Harmonic Analysis*, Heidelberg–Berlin –N. Y. 1, 1963; 2, 1970.

2. I. **Neiman**, Izv. Research Inst. of Math. and Mech. at Tomsk State Univ. named after V. V. Kuibyshev, 4, 205 (1962).

3. F. B. **Wright**, Ann. Math. (2) 60, 560 (1954).

Note: Figure translations are in progress. See original paper for figures.

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