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**Abstract**

**Full Text**

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**MATHEMATICS**

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## AN ESTIMATE OF THE SOLUTION OF THE TRICOMI PROBLEM FOR A CLASS OF EQUATIONS OF MIXED TYPE

*(Presented by Academician I. N. Vekua on 30 I 1970)*

### § 1. The equation

$$\operatorname{sgn} y \cdot |y|^m u_{xx} + u_{yy} - \lambda^2 \operatorname{sgn} y \cdot |y|^m u = 0 \quad (m > 0) \quad (1)$$

will be considered in a domain  $D$  bounded by a smooth curve  $\Gamma_0$ , lying in the half-plane  $y \geq 0$ , with endpoints at the points  $A(0,0)$ ,  $B(1,0)$ , and by the characteristics  $\Gamma_1, \Gamma_2$  of equation (1) issuing from the points  $A$  and  $B$ .

**The Tricomi problem.** In the domain  $D$ , find a solution  $u(x, y, \lambda)$  of equation (1) satisfying the boundary conditions

$$\begin{aligned} u|_{\Gamma_0} &= \varphi(x, \lambda), & 0 \leq x \leq 1, & & -\infty < \lambda < +\infty, \\ u|_{\Gamma_1} &= \psi(x, \lambda), & 0 \leq x \leq \frac{1}{2}, & & -\infty < \lambda < +\infty, \end{aligned} \quad (2)$$

and the gluing conditions

$$u|_{y=0-0} = u|_{y=0+0}, \quad \partial u / \partial y|_{y=0-0} = \partial u / \partial y|_{y=0+0}.$$

The uniqueness of the solution of the Tricomi problem for the equation

$$y u_{xx} + u_{yy} + c(x, y) u = 0 \quad (\text{F})$$

was proved in <sup>(1,2)</sup>. In <sup>(3)</sup> a principle of absolute extremum was established for equation (F) when  $c(x, y) = -\lambda^2 y$ . In all these works the proof of uniqueness of the solution of the Tricomi problem was carried out under the condition that the coefficient  $c(x, y)$  is sufficiently small, or when the length of the transition

line is sufficiently small. By means of a certain device it turns out to be possible to remove this restriction for equation (1).

Introduce a new unknown function  $v(x, y, \lambda)$  by the formula

$$u(x, y, \lambda) = v(x, y, \lambda) \exp \left[ \int \omega(x) dx \right], \quad (3)$$

where  $\omega(x)$  is a continuously differentiable and nonnegative solution on  $[0; 1]$  of the differential equation

$$d\omega/dx + \omega^2 - \lambda^2 = 0. \quad (4)$$

The function  $v(x, y, \lambda)$  is a solution of the equation

$$\operatorname{sgn} y \cdot |y|^m v_{xx} + v_{yy} + 2 \operatorname{sgn} y \cdot |y|^m \omega(x) v_x = 0. \quad (5)$$

As a function  $\omega(x)$  with the required properties one may take the function  $\omega(x) = \lambda \operatorname{th} \lambda x$ . For the coefficients of equation (5), certain conditions established in (4) are fulfilled. The function

$$z(x, y, \lambda) = \pm v(x, y, \lambda) + My,$$

where  $M$  is a constant, is also a solution of equation (5). Choose  $M$  so that  $z(x, y, \lambda)$  is a nondecreasing function of  $y$  on the characteristic

$\Gamma_1$ . For this it is enough to set

$$M = \max_{\Gamma_1} |v_y - (y)^{m/2} v_x|. \quad (6)$$

Then the function  $z(x, y, \lambda)$  satisfies all the conditions of Protter's theorem (see (4), § 5, Theorem 4), on the basis of which we can write

$$|v(x, y, \lambda)| \leq \max_{\Gamma_0} |v(x, y, \lambda)| + MM_1, \quad (7)$$

where  $M_1 = \max(y_0 - y)$ ,  $y_0 = \max_{\bar{D}} |y|$ .

Set

$$g(x) = \exp \left[ - \int \omega(x) dx \right].$$

Taking here  $\omega(x) = \lambda \operatorname{th} \lambda x$ , we obtain  $g(x) = 1/2 \operatorname{ch} \lambda x$ . Since  $v(x, y, \lambda) = g(x)u(x, y, \lambda)$ , passing in (6) to the function  $u(x, y, \lambda)$ , we find

$$M = \max_{\Gamma_1} |g(x) [u_y - (-y)^{m/2} u_x] - (-y)^{m/2} u g'(x)|.$$

Hence we obtain the estimate

$$M \leq M_2 \left[ \max_{\Gamma_1} g(x) |\psi'_x(x, \lambda)| + |\lambda| \max_{\Gamma_1} g(x) |\psi(x, \lambda)| \right], \quad (8)$$

where  $M_2 = \max_{\Gamma_1} (-y)^{m/2}$ .

Now we rewrite estimate (7), taking (8) into account, as follows:

$$\begin{aligned} |u(x, y, \lambda)| g(x) &\leq \max_{\Gamma_0} g(x) |\varphi(x, \lambda)| + \\ &+ M_1 M_2 \left[ \max_{\Gamma_1} g(x) |\psi'_x(x, \lambda)| + |\lambda| \max_{\Gamma_1} g(x) |\psi(x, \lambda)| \right]. \end{aligned}$$

Taking further into account that

$$\max_{\Gamma_1} g(x) = \max_{\Gamma_0} g(x) = 1/2,$$

we finally obtain the estimate of the solution of problem (1), (2):

$$\begin{aligned} |u(x, y, \lambda)| &\leq \operatorname{ch} \lambda x \left\{ \max_{\Gamma_0} |\varphi(x, \lambda)| + \right. \\ &\left. + M_1 M_2 \left[ \max_{\Gamma_1} |\psi'_x(x, \lambda)| + |\lambda| \max_{\Gamma_1} |\psi(x, \lambda)| \right] \right\}. \end{aligned} \quad (9)$$

From (9) follows the uniqueness of the solution of problem (1), (2), as well as the continuous dependence of this solution on the boundary functions for any fixed  $\lambda$ .

§ 2. Consider the equation

$$\operatorname{sgn} y \cdot |y|^m (u_{xx} + u_{zz}) + u_{yy} = 0, \quad m > 0. \quad (10)$$

The result obtained in § 1 makes it possible to solve the following boundary-value problem. Let  $G$  be a domain of three-dimensional space  $(x, y, z)$ , bounded by the surfaces:

$$S_0 : x(1-x) = \frac{4}{(m+2)^2} y^{m+2}, \quad y \geq 0, \quad -\infty < z < +\infty,$$

$$S_1 : x - \frac{2}{m+2}(-y)^{(m+2)/2} = 0, \quad 0 \leq x \leq 1/2, \quad -\infty < z < +\infty,$$

$$S_2 : x + \frac{2}{m+2}(-y)^{(m+2)/2} = 0, \quad 1/2 \leq x \leq 1, \quad -\infty < z < +\infty.$$

**Problem TS.** In the domain  $G$ , find a solution  $w(x, y, z)$  of equation (10), continuous in  $\overline{G}$  and satisfying the conditions:

$$w|_{S_0} = \Phi(x, z), \quad w|_{S_1} = \Psi(x, z),$$

$$\lim_{z \rightarrow \infty} w(x, y, z) = \lim_{z \rightarrow \infty} \partial w / \partial z = 0. \quad (11)$$

This problem was considered for  $m = 1$  in (3). However, in that work the existence of a solution of problem (1), (2) for arbitrary  $\lambda$  was not proved, and no conditions were obtained ensuring the existence of a solution in the form of a Fourier integral. Our estimates for the solution of problem (1), (2) make it possible to fill this gap. Let  $w(x, y, z)$  be a solution of problem TS. Its Fourier transform with respect to the variable  $z$ ,

$$u(x, y, \lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} w(x, y, z) e^{i\lambda z} dz,$$

is a solution in the domain  $D$  of equation (1), satisfying the conditions

$$u|_{\Gamma_0} = \varphi(x, \lambda), \quad -\infty < \lambda < +\infty,$$

$$u|_{\Gamma_1} = \psi(x, \lambda), \quad -\infty < \lambda < +\infty,$$

where  $D$  is the plane domain bounded by the arcs  $\Gamma_0, \Gamma_1, \Gamma_2$ , which are the lines of intersection of the plane  $z = 0$  with the surfaces  $S_0, S_1, S_2$ , respectively. Here

$$\varphi(x, \lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi(x, z) e^{i\lambda z} dz,$$

$$\psi(x, \lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x, z) e^{i\lambda z} dz.$$

Thus, problem TS is reduced to problem (1), (2). Conversely, if the function  $u(x, y, \lambda)$  is a solution of the Tricomi problem (1), (2), then the solution of problem TS is obtained by means of the inverse Fourier transform:

$$w(x, y, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x, y, \lambda) e^{-iz\lambda} d\lambda. \quad (12)$$

Taking into account the estimate (9) for the function  $u(x, y, \lambda)$ , we can impose on the boundary functions  $\Phi(x, z)$  and  $\Psi(x, z)$  conditions ensuring the existence of the integral (12). Let the functions  $\Phi(x, z)$ ,  $\Psi(x, z)$  be such that, for the functions  $\varphi(x, \lambda)$ ,  $\psi(x, \lambda)$ , the estimates

$$\varphi(x, \lambda) = O(1/|\lambda|^\alpha \operatorname{ch} \lambda), \quad \psi(x, \lambda) = O(1/|\lambda|^{1+\alpha} \operatorname{ch} \lambda),$$

$$\psi'_x(x, \lambda) = O(1/|\lambda|^\alpha \operatorname{ch} \lambda), \quad \alpha > 1.$$

hold. Then from (9) we have the estimate  $u(x, y, \lambda) = O(1/|\lambda|^\alpha)$ , which ensures the existence of the integral (12), giving a solution of problem TS.

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*Note: Figure translations are in progress. See original paper for figures.*

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