Reports of the Academy of Sciences of the USSR

THEORY OF ELASTICITY

1970

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1970, Volume 190, No. 6

UDC 539.312

THEORY OF ELASTICITY

E. F. AFANAS' EV, G. P. CHEREPANOV

A SELF-SIMILAR PROBLEM OF THE DYNAMIC THEORY OF ELASTICITY FOR A CRACK WITH A POINT SOURCE

(Presented by Academician L. I. Sedov on 3 VII 1969)

A class of self-similar dynamic problems of the plane theory of elasticity is considered for an unbounded body with an expanding rectilinear cut, free of loads at any time $t > 0$. It can be shown that only three solutions have physical meaning: a) a solution with constant finite momentum (the displacement potentials are homogeneous functions of the coordinates and time of zero dimension); b) a solution with constant finite energy (the displacements are homogeneous functions of the coordinates and time of zero dimension); c) a solution in which the stresses are homogeneous functions of the coordinates and time of zero dimension. The second problem, in its physical formulation, is analogous to Sedov' s problem of a strong explosion ([1]); the solution of the last problem was earlier obtained by Broberg ([2]) by a rather complicated method. Here a closed solution of the first two problems is given; by the Smirnov–Sobolev method ([3]) they (as well as Broberg' s problem) are reduced to the Keldysh–Sedov boundary-value problem ([4]).

Formulation of the problem. Consider an infinite homogeneous and isotropic ideally elastic space, in a state of plane strain. Suppose that in this space at the time $t = 0$ an instantaneous source concentrated at the origin acts, so that in both directions along the x-axis a cut begins to propagate with constant velocity c, the surfaces of which are free of loads. We assume that $c < c_2 < c_1$, where c_1, c_2 are the velocities of longitudinal and transverse waves.

In the plane dynamic problem of the theory of elasticity the components of displacement u and v along the Cartesian coordinate axes x and y are expressed in terms of the potentials $\varphi(x, y, t)$ and $\psi(x, y, t)$ as follows:

$$u = \partial\varphi/\partial x + \partial\psi/\partial y, \qquad v = \partial\varphi/\partial y - \partial\psi/\partial x. \tag{1}$$

The displacement vector $\mathbf{V}$ can be represented as the sum of a potential vector $\mathbf{V}_1$ and a solenoidal vector $\mathbf{V}_2$,

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2, \qquad \nabla \times \mathbf{V}_1 = 0, \qquad \nabla \cdot \mathbf{V}_2 = 0, \qquad \mathbf{V}_k = \{u_k, v_k\}. \quad (2)$$

The functions φ, u_1, v_1 satisfy the wave equation for longitudinal waves, and the functions ψ, u_2, v_2 the wave equation for transverse waves. The components of the stress tensor $\sigma_x, \sigma_y, \tau_{xy}$, according to Hooke' s law, are

$$\sigma_x = \mu\left[\frac{c_1^2}{c_2^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - 2\frac{\partial v}{\partial y}\right], \qquad \sigma_y = \mu\left[\frac{c_1^2}{c_2^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - 2\frac{\partial u}{\partial x}\right], \quad (3)$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

(μ is the Lamé constant).

The formulated problem of the cut is symmetric with respect to the x-axis; therefore it is sufficient to find the solution in the upper half-plane $y \geqslant 0$.

The boundary conditions of the problem for the half-plane $y \geqslant 0$ and any instant of time t are written in the form

$$\begin{array}{llll} \sigma_y = 0, & \tau_{xy} = 0 & \text{for} \quad y = 0, & |x| < ct, \\ v = 0, & \tau_{xy} = 0 & \text{for} \quad y = 0, & |x| > ct. \end{array} \quad (4)$$

Problem with finite impulse. Let the displacement potentials $\varphi(x, y, t)$ and $\psi(x, y, t)$ be homogeneous functions of the coordinates and time of zero dimension. In this case the representations (3) take place

$$\varphi(x, y, t) = \operatorname{Re}\Phi(z_1), \qquad \psi(x, y, t) = \operatorname{Re}\Psi(z_2); \quad (5)$$

$$z_k = \left[xt - iy\sqrt{t^2 - C_k^2(x^2 + y^2)}\right] / (x^2 + y^2) \quad (k = 1, 2), \quad (6)$$

where the radicals are understood arithmetically. Here the physical half-plane $y > 0$ corresponds to the lower half-planes of the complex variables z_k, i.e. $\operatorname{Re} z_k < 0$.

The functions $\Phi(z_1)$ and $\Psi(z_2)$ will be analytic functions of their arguments in their domains of complexness. Taking into account (1), (3), (5), and (6), one can show that the boundary conditions (4) are formulated in the form of the following boundary-value problem for the functions $\Phi(z)$ and $\Psi(z)$, analytic in the lower half-plane $\operatorname{Im} z < 0$:

$$\operatorname{Re}\left\{2z\sqrt{c_1^{-2}-z^2}\,\Phi'(z)+(c_2^{-2}-2z^2)\Psi'(z)\right\}=0 \quad \text{for } \operatorname{Im} z=0,$$

$$\operatorname{Re}\left\{\sqrt{c_1^{-2}-z^2}\,\Phi'(z)-z\Psi'(z)\right\}=0 \quad \text{for } \operatorname{Im} z=0,\ |z|<c^{-1}, \tag{7}$$

$$\operatorname{Re}\left\{(c_2^{-2}-2z^2)\Phi'(z)-2z\sqrt{c_2^{-2}-z^2}\,\Psi'(z)\right\}=0 \quad \text{for } \operatorname{Im} z=0,\ |z|>c^{-1}$$

$$(z=t/x).$$

Here, as $z\to\infty$, the radicals have order $iz+O(z^{-1})$; $\Phi'(z)$ and $\Psi'(z)$ are derivatives.

Equating identically to zero the expression standing under the sign Re in the first relation (7), and then eliminating the function $\Phi'(z)$, we arrive at the Keldysh-Sedov boundary-value problem ([4])

$$\operatorname{Re}\Psi'(z)=0 \quad \text{for } \operatorname{Im} z=0, \quad |z|<c^{-1},$$

$$\operatorname{Im}\Psi'(z)=0 \quad \text{for } \operatorname{Im} z=0, \quad |z|>c^{-1}. \tag{8}$$

The solution of the boundary-value problem (8), and together with it also of problem (7), satisfying the additional conditions of boundedness of the displacement on the cut and the symmetry conditions, has the form

$$\Phi'(z)=-A(c_2^{-2}-2z^2)\sqrt{z^2-c^{-2}}\,/\,2z^2\sqrt{c_1^{-2}-z^2},$$

$$\Psi'(z)=A\sqrt{z^2-c^{-2}}\,/\,z. \tag{9}$$

Here $\sqrt{z^2-c^{-2}}/\sqrt{c_1^{-2}-z^2}\to -i+O(x^{-1})$ as $z\to\infty$, and A is a material constant. The constant A is determined through the value I of the constant finite impulse of half of the disturbed region for $y>0$ (for the entire disturbed region the total impulse, owing to symmetry, will be equal to zero). This quantity characterizes the initial intensity of the point source under consideration,

$$\iint_S \rho\,\frac{\partial v}{\partial t}\,dx\,dy=I \quad (S:\ x^2+y^2\leqslant c_1^2t^2,\ y\geqslant 0), \tag{10}$$

where ρ is the density of the medium,

$$A = 4c_2^2 cI/\pi\rho(c_1 - c)\,[c_1(c_1 + c)^2 - 4c_2^2]. \tag{11}$$

Problem with finite energy. Suppose that at $t = 0$, at the origin, an energy of magnitude E is instantaneously released. The displacements $u(x, y, t)$ and $v(x, y, t)$ will then be homogeneous functions of their variables of zero dimension.

In this case, in the perturbed region

$$u_k(x, y, t) = \operatorname{Re} U_k(z_k), \qquad v_k(x, y, t) = \operatorname{Re} V_k(z_k) \qquad (k = 1, 2), \tag{12}$$

where U_k and V_k are analytic functions of their variables.

With the aid of (2), (3), (6), and (12), the boundary conditions (4) are reduced to a boundary-value problem for the functions $U_k(z)$ and $V_k(z)$, analytic in the lower half-plane $\operatorname{Im} z < 0$,

$$\operatorname{Re}\left\{2\sqrt{c_1^{-2} - z^2}\, U_1'(z) + \sqrt{c_2^{-2} - z^2}\, U_2'(z) + zV_2'(z)\right\} = 0 \quad \text{for } \operatorname{Im} z = 0,$$

$$\operatorname{Re}\left\{\sqrt{c_1^{-2} - z^2}\, U_1'(z) - zV_1'(z)\right\} = 0 \quad \text{for } \operatorname{Im} z = 0, \tag{13}$$

$$\operatorname{Re}\left\{zU_2'(z) + \sqrt{c_2^{-2} - z^2}\, V_2'(z)\right\} = 0 \quad \text{for } \operatorname{Im} z = 0,$$

$$\operatorname{Re}\left\{\frac{c_1^2}{c_2^2}\left[zU_1'(z) + \sqrt{c_1^{-2} - z^2}\, V_1'(z)\right] - 2z\left[U_1'(z) + U_2'(z)\right]\right\} = 0$$

$$\text{for } \operatorname{Im} z = 0, \quad |z| < c^{-1},$$

$$\operatorname{Re}[V_1'(z) + V_2'(z)] = 0 \quad \text{for } \operatorname{Im} z = 0, \quad |z| > c^{-1}$$

$$(z = t/x).$$

Problem (13) can be reduced to the Keldysh–Sedov boundary-value problem for the function $V_2(z)$,

$$\operatorname{Re} V_2'(z) = 0 \quad \text{for } \operatorname{Im} z = 0, \quad |z| < c^{-1},$$

$$\operatorname{Im} V_2'(z) = 0 \quad \text{for } \operatorname{Im} z = 0, \quad |z| > c^{-1}. \tag{14}$$

The solution of problem (14), and together with it of problem (13), satisfying the additional physical conditions of boundedness of the displacement on the cut and the symmetry conditions, can be represented in the form

$$U_1'(z) = B\,\frac{c_2^{-2} - 2z^2}{2z\sqrt{c_1^{-2} - z^2}\sqrt{z^2 - c^{-2}}}, \qquad U_2'(z) = -B\,\frac{\sqrt{c_2^{-2} - z^2}}{z\sqrt{z^2 - c^{-2}}},$$

$$V_1'(z) = B\,\frac{c_2^{-2} - 2z^2}{2z^2\sqrt{z^2 - c^{-2}}}, \qquad V_2'(z) = B\,\frac{1}{\sqrt{z^2 - c^{-2}}}, \tag{15}$$

where B is a real constant. To determine it we use the energy conservation equation, which in the case under consideration is reduced to the form

$$\iint_{\xi^2+\eta^2\le c_1^2} \left\{ \frac{1}{2}(u^2+v^2) + c_1^2\left(\frac{\partial u_1}{\partial \xi} + \frac{\partial v_1}{\partial \eta}\right)^2 + c_2^2\left(\frac{\partial v_2}{\partial \xi} - \frac{\partial u_2}{\partial \eta}\right)^2 \right\} d\xi\, d\eta = \frac{E}{\rho} \tag{16}$$

$$(\xi = x/t, \qquad \eta = y/t).$$

Taking into account (2), (6), (12), and (15), from (16), after cumbersome calculations, we find

$$B = \frac{1}{2c_1}\,[E/\rho J(\alpha,\beta)]^{1/2}. \tag{17}$$

The function $J(\alpha,\beta)$, occurring in (17) and depending on two parameters $\alpha = c_2/c_1$ and $\beta = c/c_2$, is expressed in terms of quadratures

$$J(\alpha,\beta) = \int_0^{\pi/2} d\varphi \int_0^1 \left\{ \frac{1}{2}\left[M_1^2(r,\varphi) + N_1^2(r,\varphi)\right] + \frac{\alpha^2}{2}\left[M_2^2(r,\varphi) + N_2^2(r,\varphi)\right] + \right.$$

$$\left. + \alpha^2[M_1(\alpha r,\varphi)N_2(r,\varphi) + M_2(r,\varphi)N_1(\alpha r,\varphi)] + L_1^2(r,\varphi) + \alpha^2 L_2^2(r,\varphi) \right\} r\,dr.$$

Here

$$M_1 = \frac{1}{2}\beta F_1^{(1)} - F_2^{(1)}, \qquad N_1 = F - F_0^{(1)}, \qquad M_2 = -\beta F_1^{(2)} + F_2^{(2)}, \qquad N_2 = F_0^{(2)},$$

$$L_1 = \{(\gamma_1 R_0^- + \gamma_2 R_0^+)(\sqrt{1-r^2}\,R_1^+ + r^2 R_1^- \sin\varphi\cos\varphi) -$$

$$-(\gamma_1 R_0^+ - \gamma_2 R_0^-)(\sqrt{1-r^2}\, R_1^- - r^2 R_1^+ \sin\varphi\cos\varphi)\} \times \{\sqrt{1-r^2}\,[(R_0^+)^2 +$$

$$+(R_0^-)^2][(R_1^+)^2 + (R_1^-)^2]\}^{-1},$$

$$L_2 = \sqrt{2r^2}\,(\sqrt{1-r^2}\, R_2^+ + r^2 R_2^- \sin\varphi\cos\varphi) \times \{\sqrt{1-r^2}\,[(R_2^+)^2 + (R_2^-)^2]\}^{-1},$$

$$F(r,\varphi) = [n\cos\varphi + \sqrt{1-r^2}\sin\varphi\,(R_1^-)^2] \times \{\sqrt{2\rho_1}\, R_1^-\}^{-1},$$

$$F_0^{(k)}(r,\varphi) = \frac{1}{2}\ln\{[(R_k^- + \sqrt{2}\cos\varphi)^2 +$$

$$+(R_k^+ - \sqrt{2}\sqrt{1-r^2}\sin\varphi)^2]/2a_k^2 r^2\},$$

$$F_1^{(k)}(r,\varphi) = \frac{1}{2}\operatorname{arc\,tg}\{[2a_k^2 n\rho_1^2 r^2 + a_k[\rho_2(R_0^- R_k^+ + R_0^+ R_k^-) +$$

$$+n(R_0^- R_k^- - R_0^+ R_k^+)]] \times [[\rho_1^2[2a_k^2\rho_2 r^2 - (a_k^2+1)]] +$$

$$+a_k[\rho_2(R_0^- R_k^- + R_0^+ R_k^+) - n(R_0^- R_k^+ + R_0^+ R_k^-)]]^{-1}\},$$

$$F_2^{(k)}(r,\varphi) = \frac{1}{2}\operatorname{arc\,tg}\{\{[2n + (R_0^- R_k^+ + R_0^+ R_k^-)]\}^{-1}[(a_k^2+1)r^2 - 2\rho_2 +$$

$$+(R_0^- R_k^- - R_0^+ R_k^+)]^{-1}\},$$

$$\gamma_1 = \frac{r^2 - 2a^2\rho_1}{a^2\rho_1^2}\sin\varphi, \qquad \gamma_2 = \frac{2\sqrt{1-r^2}\,(r^2 + 2a^2\rho_1)}{a^2\rho_1 r}\cos\varphi, \qquad \rho_1 = 1 - r^2\sin^2\varphi,$$

$$\rho_2 = \cos^2\varphi - (1-r^2)\sin^2\varphi, \qquad R_k^\pm = \sqrt{\sqrt{m_k^2 + n^2} \pm m_k},$$

$$n = 2\sqrt{1-r^2}\sin\varphi\cos\varphi, \qquad m_k = \rho_1 - a_k^2 r^2, \qquad a_0 = 1, \qquad a_1 = 1/\alpha\beta, \qquad a_2 = 1/\beta.$$

Stress-intensity coefficient. For fracture mechanics, the stress field near the end of a slit is of principal interest; in the present case it is described by the coefficient

$$K_I = \lim_{x-ct\to 0}[\sigma_y\sqrt{2\pi(x-ct)}] = \frac{\mu\sqrt{2\pi}}{2-2\nu}\lim_{ct-x\to 0}\frac{v}{\sqrt{ct-x}} \tag{18}$$

(ν is Poisson' s ratio).

From the solutions obtained we compute the stress-intensity coefficient:

solution with finite impulse

$$K_I = \mu A\sqrt{\pi}/t^{1/2}c_2^2\sqrt{c}\,(2-2\nu), \tag{19}$$

solution with finite energy

$$K_I = \mu Bc^{3/2}\sqrt{\pi}/\sqrt{t}\,c_2^2(2-2\nu). \tag{20}$$

The exact solutions obtained can be used for an approximate solution of the problem of the size of a crack produced as a result of a dynamic action. In the case of an ideally brittle body, it is then necessary to use an additional boundary condition on the contour of the dynamic crack (5).

Moscow Institute
of the Petrochemical and Gas Industry
named after N. M. Gubkin

Received
18 VI 1969

CITED LITERATURE

1. L. I. Sedov, *Similarity and Dimensional Methods in Mechanics*, Moscow, 1957.

2. K. B. Broberg, Ark. Fys., **18**, 159 (1960).

3. V. Smirnoff, S. Soboleff, Transactions of the Seismological Institute, No. 20 (1931).

4. M. A. Lavrent' ev, B. V. Shabat, *Methods of the Theory of Functions of a Complex Variable*, Moscow-Leningrad, 1958.

5. G. P. Cherepanov, Int. J. Solids Structures, **4** (1968).