ON THE EXPANSION OF FUNCTIONS OF THE CLASS $B^{\alpha}_{p,\theta}$ IN A FOURIER SERIES WITH RESPECT TO AN ARBITRARY FUNDAMENTAL SYSTEM OF FUNCTIONS OF THE LAPLACE OPERATOR

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Abstract

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Sh. A. Alimov

ON THE EXPANSION OF FUNCTIONS OF THE CLASS $B^{\alpha}_{p,\theta}$ IN A FOURIER SERIES WITH RESPECT TO AN ARBITRARY FUNDAMENTAL SYSTEM OF FUNCTIONS OF THE LAPLACE OPERATOR

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Consider an arbitrary N-dimensional domain g, star-shaped with respect to some ball, and an arbitrary fundamental system of functions (f.s.f.) $\{u_n(x)\}$ of the Laplace operator in this domain (i.e., a complete orthonormal system of eigenfunctions of an arbitrary nonnegative self-adjoint extension of the Laplace operator in the domain g, with spectrum consisting of eigenvalues $\lambda_n \geq 0$).

The present work is devoted to the study of expansions of functions from the class $B^{\alpha}_{p,\theta}$ in a Fourier series with respect to the system $\{u_n(x)\}$.* In what follows, by $\overset{\circ}{B}{}^{\alpha}_{p,\theta}(g)$ we shall denote the set of functions $f(x)$ belonging in the whole space to the Besov class $B^{\alpha}_{p,\theta}(E_N)$ and equal to zero outside the domain g.

For an arbitrary function $f(x) \in L_2(g)$, with Fourier coefficients f_n with respect to the system $\{u_n(x)\}$, and for any $a > 0$, define the quantity

$$\rho_a(f) = \left(\sum_{a < \sqrt{\lambda_n} \leq 2a} f_n^2 \right)^{1/2}. \qquad (1)$$

The following theorem on the Fourier coefficients of functions $f(x) \in \overset{\circ}{B}{}^{\alpha}_{2,\theta}(g)$ is valid.

Theorem 1. Let $\alpha > 0$ and let θ satisfy the condition $1 \geq \theta \leq \infty$. Then, for an arbitrary function $f(x) \in \overset{\circ}{B}{}^{\alpha}_{2,\theta}(g)$, the inequality

$$\left[\sum_{m=0}^{\infty}\left(2^{m\alpha}\rho_{2^m}(f)\right)^{\theta}\right]^{1/\theta}\leq c\|f\|_{B^{\alpha}_{2,\theta}} \tag{2}$$

holds.

Let us briefly discuss the scheme of the proof of the theorem. For any vector u and any number m, introduce the notation

$$\Delta_u^m f(x)=\sum_{k=0}^{m}(-1)^{m+k}C_m^k f(x+ku). \tag{3}$$

For $h>0$, define the modulus of continuity of order m by

$$\omega_m(f,h)=\sup_{|u|\leq h}\|\Delta_u^m f(x)\|_{L_2(E_N)}.$$

Choose a spherical coordinate system (r,θ) with origin at the center of the ball with respect to which the domain g is star-shaped, and introduce the notation

$$\widetilde{\Delta}_h^m f(x)=\sum_{k=0}^{m}(-1)^{m+k}C_m^k f(r+kh,\theta).$$

* For the definition of the Besov classes $B^{\alpha}_{p,\theta}$, as well as the Nikol' skii classes H^{α}_p and Liouville classes L^{α}_p, see (1).

Define the spherical modulus of continuity of order m

$$\tilde{\omega}_m(f,h)=\sup_{0<t<h}\left\|\widetilde{\Delta}_t^m f(x)\right\|_{L_2(E_N)}.$$

For any number m and $t\geq 0$, introduce the function

$$\psi_m(t)=2^{N/2-1}\Gamma\left(\frac{N}{2}\right)\sum_{k=0}^{m}(-1)^{m+k}C_m^k(kt)^{1-N/2}J_{N/2-1}(kt).$$

Applying the mean-value formula to the function (3) and using Parseval' s equality, we obtain the principal estimate

$$\left[\sum_{n=1}^{\infty}\varphi_m^2\left(h\sqrt{\lambda_n}\right)f_n^2\right]^{1/2}\leq c\left[\omega_m(f,h)+\tilde{\omega}_m(t,h)\right]. \tag{4}$$

From the membership of the function $f(x)$ in the class $B^{\alpha}_{2,\theta}$, for $\omega_m(f,h)$ there follows the estimate $(m>\alpha)$

$$\left(\int_0^\infty \left(h^{-\alpha}\omega_m(f,h)\right)^\theta \frac{dh}{h} \right)^{1/\theta} \le c\|f\|_{B^\alpha_{2,\theta}}. \tag{5}$$

Representing $f(x) \in B^\alpha_{2,\theta}$ in the form of a series whose terms are functions of exponential type, it is not difficult to verify the validity of estimate (5) also for $\tilde{\omega}_m(f,h)$ (see ([1]), p. 260). Applying this estimate to inequality (4), we arrive at estimate (2).

Let us note that, for a function $f(x)$ finite in an arbitrary strictly interior subdomain g' of the domain g, when $\theta \ge 2$, by means of the same arguments one can verify the validity of the reverse inequality

$$\|f\|_{B^\alpha_{2,\theta}} \le c(g') \left[\sum_{m=1}^\infty \left(2^{m\alpha}\rho_{2^m}(f)\right)^\theta \right]^{1/\theta}.$$

In the case when the domain g is an N-dimensional cube with side 2π, and $\{u_n(x)\}$ is the multiple trigonometric system, this result, as well as estimate (2), is known (see ([1]), Theorem 8.10.1).

1°. We give some consequences of Theorem 1. Let us first formulate it for the cases $\theta = \infty$ and $\theta = 2$.

Corollary 1. Let $f(x) \in \dot{H}^\alpha_2(g)$, $\alpha > 0$. Then, for every $\mu > 0$, the inequality

$$\sum_{\mu < \sqrt{\lambda_n} \le 2\mu} f_n^2 \lambda_n^\alpha \le c\|f\|^2_{\dot{H}^\alpha_2} \tag{6}$$

is valid.

Corollary 2. For any function $f(x) \in L^\alpha_2(g)$, with $\alpha \ge 0$, the inequality

$$\sum_{n=1}^\infty f_n^2 \lambda_n^\alpha \le c\|f\|^2_{L^\alpha_2}$$

is valid.

For $\theta = 1$, the following assertion follows from Theorem 1.

Theorem 2. Let p satisfy the condition $1 \le p \le 2$. Then the Fourier series of an arbitrary function $f(x) \in \dot{B}^{N/p}_{p,1}$ converges uniformly and absolutely in any strictly interior subdomain g' of the domain g.

For the proof we use the Cauchy–Bunyakovsky inequality and definition (1):

$$\sum_{n=1}^{\infty} |f_n u_n(x)| \leq \sum_{m=0}^{\infty} \rho_{2^m}(f) \left[\sum_{\sqrt{\lambda_n} < 2^m} u_n^2(x) \right]^{1/2}. \tag{7}$$

Let us further note that, uniformly in x belonging to g', the estimate

$$\sum_{\sqrt{\lambda_n} < \mu} u_n^2(x) \leq c\mu^N.$$

It remains to substitute this estimate into inequality (7) and apply estimate (2) with $\theta = 1$.

Corollary. *The Fourier series of an arbitrary function $f(x) \in \dot{W}_1^N(g)$ converges uniformly and absolutely in any strictly interior subdomain g' of the domain g.*

Let us note that Theorem 2 is final. For an arbitrary F.S.F. of the Laplace operator in any domain g and any interior point $x_0 \in g$ there exists a finite function $f(x)$, belonging to the class $B_{p,\theta}^{N/p}$ for any $p \geq 1$ and any $\theta > 1$, whose Fourier series diverges at the point x_0. Moreover, under the same conditions one can indicate a finite function $f(x)$, for any $p > 2$ belonging to the class $B_{p,1}^{N/p}$, whose Fourier series diverges absolutely at the point x_0.

2°. Let us consider in more detail the case $N = 2$, when g is a two-dimensional domain star-shaped with respect to some circle. From the results of [2] it follows that, for localization of the partial sums of the Fourier series of an arbitrary function $f(x) \in L_2(g)$, it is sufficient that the quantity standing on the left-hand side of inequality (6), with $\alpha = 1/2$, be uniformly bounded with respect to μ. Thus we obtain that a sufficient condition for localization is that the function $f(x)$ belong to the class $\dot{H}_2^{1/2}(g)$.

We shall say that a function $f(x)$ is piecewise continuous in the domain g if this domain can be divided, by means of rectifiable curves, into a finite number of domains g_k, in each of which $f(x)$ is uniformly continuous. If, in addition, $f(x)$ has uniformly continuous first-order derivatives in each of the domains g_k, then the function $f(x)$ will be called piecewise smooth. It is not difficult to show that every piecewise smooth function belongs to the class $\dot{H}_2^{1/2}(g)$. Thus, for localization of the partial sums of the Fourier series in the case $N = 2$, there is no need to require that the function $f(x)$ satisfy any boundary conditions. Let us note that in the case $N > 2$, even for the function identically equal to one in the N-dimensional ball, the Fourier series with respect to the eigenfunctions of the first boundary-value problem diverges at the center of the ball.

It can also be shown that any function belonging, in a two-dimensional domain g star-shaped with respect to some circle, to the space $W_2^1(g)$, also belongs to the class $\dot{H}_2^{1/2}(g)$. In connection with this, let us consider in the domain g with boundary Γ the Dirichlet problem

$$\Delta u = 0, \qquad x \in g, \qquad u|_\Gamma = \varphi \tag{8}$$

with an arbitrary admissible function φ.

Since the generalized solution $u(x) \in W_2^1(g)$, it follows that $u(x)$ belongs to the class $\dot{H}_2^{1/2}(g)$, and the following assertion is valid.

Theorem 3. *Let g be an arbitrary two-dimensional domain, star-shaped with respect to some circle. Then for any function $u(x)$ that is a generalized solution of the Dirichlet problem in the domain g, the Fourier series of $u(x)$ with respect to an arbitrary F.S.F. of the Laplace operator converges uniformly in any strictly interior subdomain g' of the domain g.*

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.