THE MILNE PROBLEM FOR POLARIZED RADIATION WITH THE RAYLEIGH SCATTERING LAW

MATHEMATICAL PHYSICS

1970

SovietRxiv

Abstract

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UDC 535.51

MATHEMATICAL PHYSICS

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THE MILNE PROBLEM FOR POLARIZED RADIATION WITH THE RAYLEIGH SCATTERING LAW

(Presented by Academician A. N. Tikhonov on 8 I 1970)

We consider the problem of the transfer of polarized radiation in a semi-infinite plane atmosphere $z > 0$, scattering according to Rayleigh' s law, under the conditions of constancy of the total flux and absence of external radiation incident on the boundary $z = 0$. It is known that the problem admits an azimuthally homogeneous solution corresponding to a field of linearly polarized radiation, the plane of polarization coinciding with the meridional plane determined by the unit vector $\mathbf{n}$ of the inward normal to the surface $z = 0$ and the unit vector ω of the direction of propagation of the radiation at each point. Such a radiation field is completely described by the two-dimensional vector

$$\Psi(z,\mu) = \begin{bmatrix} \Psi_1(z,\mu) \\ \Psi_2(z,\mu) \end{bmatrix}, \tag{1}$$

where $\Psi_1(z,\mu)$ and $\Psi_2(z,\mu)$ are the intensity components of the radiation with mutually orthogonal states of polarization; z is the optical thickness, measured in units of the mean free path for Rayleigh scattering; $\mu = (\mathbf{n}, \omega)$. The problem is of interest for the atmospheres of early-type stars, in which the only mechanism of interaction of radiation with matter is Thomson scattering by free electrons.

The function $\Psi(z,\mu)$ satisfies the transfer equation

$$\mu \frac{\partial \Psi(z,\mu)}{\partial z} + \Psi(z,\mu) = \frac{1}{2} \int_{-1}^{1} \hat{K}(\mu,\mu') \Psi(z,\mu')\, d\mu', \tag{2}$$

where

$$\hat{K}(\mu,\mu') = \frac{3}{4} \begin{bmatrix} 2(1-\mu^2)(1-\mu'^2) + \mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{bmatrix}, \tag{3}$$

with the boundary conditions

$$\Psi(0,\mu) = 0, \qquad \mu > 0; \tag{4}$$

$$\lim_{z\to\infty} e^{-z}\Psi(z,\mu) = 0. \tag{5}$$

Introduce the moments of the function $\Psi(z,\mu)$

$$\Psi^k(z) = \int_{-1}^{1} \Psi(z,\mu) P_k(\mu)\, d\mu, \tag{6}$$

where $P_k(\mu)$ is the Legendre polynomial of order k, and put, moreover,

$$\mathbf{F}(z) \equiv \Psi^1(z), \qquad \mathbf{K}(z) \equiv \frac{1}{3}\{2\Psi^2(z) + \Psi^0(z)\}. \tag{7}$$

Representing $\hat{K}(\mu,\mu')$ in the form

$$\hat{K}(\mu,\mu') = \hat{A}_1 + \hat{A}_3 P_2(\mu) + \{\hat{A}_2 + \hat{A}_4 P_2(\mu)\} P_2(\mu'), \tag{8}$$

where $\hat{A}_k,\ k = 1,\dots,4$, are numerical matrices, it is not difficult to obtain for the moments $\Psi^k(z),\ k = 0,1,2$, a system of ordinary differential equa-

$$d\mathbf{F}/dz = (\hat{A}_1 - \hat{I})\Psi^0(z) + \hat{A}_2\Psi^2(z), \tag{9}$$

$$d\mathbf{K}/dz = -\mathbf{F}(z), \tag{10}$$

which follows from equation (2). (The symbol $\hat{I}$ denotes the unit matrix of second order.) From (9) and (10) the conservation laws follow:

$$F = F_1(z) + F_2(z) = \text{const}, \tag{11}$$

$$K(z) = K_1(z) + K_2(z) = -Fz + K(0). \tag{12}$$

The function $\Psi(z,\mu)$ is uniquely determined by its first three moments. Inverting the differential operator in the left-hand side of (2) and using equations (9) and (10), one can obtain for it the expression

$$\Psi(z,\mu) = F(z,\mu)\begin{bmatrix} 3\mu^2 - 2 \\ 1 \end{bmatrix} - \frac{\mu^2 - 1}{2\mu}\,\hat{\Lambda}_2(\mu)\int_z^{0,\infty} \Psi^0(t) e^{-(z-t)/\mu}\, dt, \tag{13}$$

where

$$F(z,\mu) = \frac{9}{2(4-3\mu^2)}\{\mu(F_1(z) - \varepsilon F_1(0)e^{-z/\mu}) + K_1(z) - \varepsilon K_1(0)e^{-z/\mu}\},$$

$$\varepsilon = \begin{cases} 1, & \mu > 0, \\ 0, & \mu < 0, \end{cases} \tag{14}$$

$$\hat{\Lambda}_2(\mu) = \begin{bmatrix} \frac{3}{2} & \frac{9}{2}\frac{\mu^2}{4-3\mu^2} \\ 0 & \frac{3}{4-3\mu^2} \end{bmatrix}. \tag{15}$$

(The upper limit of the integral in (13) is 0 for $\mu > 0$ and ∞ for $\mu < 0$.)

We now introduce the Laplace transform of the function $\Psi^k(z)$

$$\Phi^k(s) = \int_0^\infty e^{-sz}\Psi^k(z)\,dz. \tag{16}$$

Using equations (9) and (10), it is not difficult to find the relation between $\Phi^2(s)$ and $\Phi^0(s)$:

$$\Phi^2(s) = \hat{C}_1(s)[\mathbf{F}(0) - s\mathbf{K}(0)] + \hat{C}_2(s)\Phi^0(s), \tag{17}$$

where $\hat{C}_k(s)$, $k = 1, 2$, are known matrices. Applying the Laplace transform to equation (2), then multiplying it by $(1+s\mu)^{-1}$ and integrating with respect to μ over the interval $[-1, 1]$, after substituting expression (17) for $\Phi^2(s)$, we arrive at an integral equation for $\Phi^0(s)$

$$\{\hat{I} - \hat{\Lambda}(s)\}\Phi^0(s) = \mathbf{T}(s), \tag{18}$$

where

$$\hat{\Lambda}(s) = \begin{bmatrix} L_0(s) - L_2(s) & \frac{1}{4s^2-3}\{s^2 L_0(s) + (2s^2-3)L_2(s)\} \\ \frac{3(s^2-1)}{4s^2-3}L_0(s) & \end{bmatrix}, \tag{19}$$

$$\mathbf{T}(s) = \begin{bmatrix} G_1(s) + \frac{3}{4s^2-3}\{L_0(s) - 2L_2(s)\}\{F_1(0) - sK_1(0)\} \\ G_2(s) - \frac{3}{4s^2-3}L_0(s)\{F_1(0) - sK_1(0)\} \end{bmatrix}, \tag{20}$$

$$L_0(s) = \frac{1}{2s}\ln\frac{1+s}{1-s}, \qquad L_2(s) = \frac{3-s^2}{2s^2}L_0(s) - \frac{3}{2s^2}, \tag{21}$$

$$\mathbf{G}(s) = \int_{-1}^{0}\frac{\mu\Psi(0,\mu)\,d\mu}{1+s\mu}. \tag{22}$$

Owing to the triangularity of the matrix $\hat{\Lambda}(s)$, the vector equation (18) can be reduced to two independent scalar equations. Indeed, let

$$\Phi_1(s) = (s^2-1)\Phi_1^0(s)+F_1(0)-sK_1(0)+\frac{3}{4s^2-3}\{(s^2-1)\Phi_2^0(s)-F_1(0)+sK_1(0)\}, \tag{23}$$

$$\Phi_2(s) = \frac{3}{4s^2-3}\{(s^2-1)\Phi_2^0(s) - F_1(0) + sK_1(0)\}. \tag{24}$$

Then equation (18) splits into two independent equations for the functions $\Phi_1(s)$ and $\Phi_2(s)$:

$$\Omega_1(s)\Phi_1(s) = H_1(s), \tag{25}$$

$$\Omega_2(s)\Phi_2(s) = H_2(s), \tag{26}$$

where

$$\Omega_1(s) = \frac{1}{2s^2}\{(2s^2-3) - 3(s^2-1)L_0(s)\}, \tag{27}$$

$$\Omega_2(s) = \frac{1}{4}\{4s^2 - 3 - 3(s^2-1)L_0(s)\}, \tag{28}$$

$$H_1(s) = (s^2-1)G_1(s) + F_1(0) - sK_1(0), \tag{29}$$

$$H_2(s) = (s^2-1)G_2(s) - F_1(0) + sK_1(0). \tag{30}$$

Equations (25) and (26) make it possible to study the analytic structure of $\Phi^0(s)$. It turns out that in the strip $|\operatorname{Re} s| < 1$ the only singularity of $\Phi^0(s)$ is the point $s = 0$ (a pole of second order), while at the points $s = \pm 1$ the function $(s^2-1)\Phi^0(s)$ vanishes.

The analytic properties of the functions $\Omega_k(s)$, $\Phi_k(s)$, and $H_k(s)$ allow one to apply the Wiener-Hopf method to equations (25) and (26). The solutions of these equations are obtained in the form

$$\Phi_1(s) = -\frac{\sqrt{2}}{2}\frac{\tau_{2-}(1)}{\tau_{1-}(1)}F(s+1)\tau_{1-}(s), \tag{31}$$

$$\Phi_2(s) = -\sqrt{2}Fs^{-2}(s+1)\tau_{2-}(s), \tag{32}$$

$$\tau_1(s) = \frac{s^2-1}{s^2}\Omega_1(s), \qquad \tau_2(s) = s^{-2}\Omega_2(s), \tag{33}$$

$$\tau_{k-}(s) = \frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty}\frac{\ln\tau_k(u)\,du}{u-s}, \qquad 0<\beta<1, \qquad k=1,2, \tag{34}$$

$$\tau_{1-}(1) = \exp\left\{\pi^{-1}\int_0^{\pi/2}\ln\left[2\sin^4 x/(3-\sin^2 x - 3x\operatorname{ctg} x)\right]\,dx\right\}, \tag{35}$$

$$\tau_{2-}(1) = \exp\left\{\pi^{-1}\int_0^{\pi/2}\ln\left[4\sin^2 x/(3+\sin^2 x - 3x\operatorname{ctg} x)\right]\,dx\right\}. \tag{36}$$

The numerical values of the coefficients in (31) and (32) are found with the aid of a priori information about the analytic properties of $\Phi^0(s)$.

Now determining $\Phi_1^0(s)$ and $\Phi_2^0(s)$ from relations (23) and (24) and carrying out the inverse Laplace transform, we obtain exact expressions for the components of the moment $\Psi^0(z)$

$$\Psi_1^0(z) = -\frac{3}{2}F(z+z_0)+$$

$$+6\sqrt{2}\frac{\tau_{2-}(1)}{\tau_{1-}(1)}F\int_0^1\frac{e^{-z/\mu}\,d\mu}{(1+\mu)\tau_{1-}(\mu^{-1})\left[2^4\Omega_1^2(\mu^{-1})+3^2\pi^2\mu^2(1-\mu^2)^2\right]}-$$

$$-18\sqrt{2}F\int_0^1\frac{(1-\mu^2)\mu^2e^{-z/\mu}\,d\mu}{\tau_{2-}(\mu^{-1})\left[2^6\tau_2^2(\mu^{-1})+3^2\pi^2\mu^2(1-\mu^2)^2\right]}, \tag{37}$$

$$\Psi_2^0(z) = -\frac{3}{2}F(z+z_0)+6\sqrt{2}F\int_0^1\frac{(1-\mu)(4-3\mu^2)e^{-z/\mu}\,d\mu}{\tau_{2-}(\mu^{-1})\left[26\tau_2^2(\mu^{-1})+32\pi^2\mu^2(1-\mu^2)^2\right]}, \tag{38}$$

where the extrapolated length z_0 is determined by the formula

$$z_0 = 1 + \tau'_{2-}(0)/\tau_{2-}(0) = 1 + \frac{3}{\pi}\int_0^{\pi/2} \frac{3\,\mathrm{ctg}^2\,x(1 - x\,\mathrm{ctg}\,x) + x\,\mathrm{ctg}\,x}{1 + \sin^2 x - 3x\,\mathrm{ctg}\,x}\,dx. \quad (39)$$

The angular distribution of the outgoing radiation is obtained from (13) for $z = 0,\ \mu < 0$

$$\Psi(0,\mu) = F(0,\mu)\begin{bmatrix}3\mu^2 - 2\\ 1\end{bmatrix} - \frac{\mu^2 - 1}{2\mu}\,\hat{\Lambda}_2(\mu)\Phi^0(-\mu^{-1}). \quad (40)$$

Carrying out the calculations in (40), we find

$$\Psi_1(0,\mu') = -\frac{3\sqrt{2}}{8}F\,\frac{\tau_{2-}(1)}{\tau_{1-}(1)}(1+\mu')\tau_{1-}\left(\frac{1}{\mu'}\right), \quad (41)$$

$$\Psi_2(0,\mu') = -\frac{3\sqrt{2}}{8}F\tau_{2-}(1)(1+\mu')\tau_{2-}\left(\frac{1}{\mu'}\right), \qquad \mu' = -\mu > 0, \quad (42)$$

where

$$\tau_{1-}(\mu^{-1}) = \exp\left\{\frac{\mu}{\pi}\int_0^{\pi/2}\frac{\ln\left[2\sin^4 x/(3 - \sin^2 x - 3x\,\mathrm{ctg}\,x)\right]}{1 - (1-\mu^2)\sin^2 x}\,dx\right\}, \quad (43)$$

$$\tau_{2-}(-\mu^{-1}) = \exp\left\{\frac{\mu}{\pi}\int_0^{\pi/2}\frac{\ln\left[4\sin^2 x/(3 + \sin^2 x - 3x\,\mathrm{ctg}\,x)\right]}{1 - (1-\mu^2)\sin^2 x}\,dx\right\}. \quad (44)$$

Problem (2), (4), (5) was first studied by S. Chandrasekhar ([1–3]). He found an approximate solution by the method of discrete ordinates and, by passing to the limit, obtained an exact formula for the angular distribution of the outgoing radiation. Recently, a number of authors have investigated equation (2) by Case' s method ([4,5]) and by V. V. Sobolev' s method*. In particular, a system of regular and singular eigenfunctions of the characteristic equation of transport theory has been constructed, and the solution of the Milne problem has been obtained in the form of an expansion in the functions of this system. The angular distribution of the outgoing radiation is expressed in terms of Chandrasekhar' s H-functions.

The author thanks M. V. Maslennikov for discussion of the work and useful comments.

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Received
30 XII 1969

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* H. Domke, private communication.

Note: Figure translations are in progress. See original paper for figures.