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Abstract

Full Text

Mathematics

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On the Order of the Degree of Exactness of Chebyshev’ s Quadrature Formula

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I. Denote by M_n the **degree of exactness** of Chebyshev’ s quadrature formula with n nodes

$$\int_{-1}^{1} p(x)f(x)\,dx = \frac{1}{n}\sum_{k=1}^{n} f(x_k), \qquad -1 < x_1 < x_2 < \cdots < x_n < 1, \tag{1}$$

i.e., the highest degree of a polynomial for which it is valid; the weight function $p(x) \geq 0$ is assumed summable on the interval $[-1,+1]$.

In the present note we set ourselves the goal of finding estimates for the order, relative to n, of the quantity M_n, assuming that it increases without bound together with n; here we use methods of S. N. Bernstein ([1]) and N. I. Akhiezer ([2]).

II. Introduce the notation

$$t(\theta) = \pi p(-\cos\theta)|\sin\theta|, \qquad -\pi \leq \theta \leq \pi, \qquad x = -\cos\theta, \tag{2}$$

and impose on the behavior of the function $t(\theta)$ the following restrictions of a local character:

A. The function $t(\theta)$ is continuous at the point $\theta = 0$.

B. On the interval $[0,\varepsilon]$, where $\varepsilon > 0$ is a fixed small quantity, the function $t(\theta)$ is positive almost everywhere.

C. The function $t(\theta)$ does not decrease on the interval $[0,\varepsilon]$.

By $2m-1$ denote the greatest odd number not exceeding M_n, and we shall assume m so large that

$$0 < \theta_1^{(m)} < \theta_2^{(m)} < \varepsilon;$$

here $\{x_k^{(m)} = -\cos\theta_k^{(m)}\}_1^m$ are the roots of the polynomial of degree m, orthogonal on the interval $[-1,+1]$ with respect to the weight function $p(x)$.

The inequality

$$\frac{2m}{n} \le \frac{2m\theta_2^{(m)}}{\pi} t(\theta_2^{(m)}); \tag{3}$$

holds; if, moreover, the function $t(\theta)$ is continuous on $[0, \varepsilon]$, then *

$$\frac{2m}{n} \le C_1 \omega_2 \left(\frac{1}{m}; t\right) + \gamma_m \left| t(\theta_1^{(m)}) - t(0) \right| + \gamma_m t(0), \tag{4}$$

where $\omega_2(\delta; t)$ is the modulus of smoothness of the function $t(\theta)$ on the interval $[0, \varepsilon]$, and

$$\frac{8}{3} \le \gamma_m = \frac{4(2m^2+1)}{3m^2} \le 4.$$

Hence there follows a very simple result: if $t(0) < 3/8$, then

$$\overline{\lim}_{n\to\infty} M_n/n < 1;$$

thus, the condition $t(0) \ge 3/8$ is necessary (but not sufficient) in order that, for unboundedly increasing values of n, one have $M_n = n$.

* $C,\ C_1,\ C_2, \ldots$ are constants independent of m and n.

III. In all that follows we shall assume that $t(0) = 0$; in order to estimate the order of the quantity M_n, we need to find an estimate for the quantity $\theta_2^{(m)}$ for unboundedly increasing values of m.

Theorem. There is an estimate $\theta_2^{(m)} \le C\delta_m$, where the quantity δ_m can be found as the root of the equation*

$$\frac{1}{\delta} \lg \frac{2c_0}{a(\delta; t)} = m, \qquad c_0 = \frac{1}{\pi} \int_0^\pi t(\theta)\, d\theta; \tag{5}$$

by $a(\delta; t)$ is denoted the modulus of growth (see (4)) of the function $t(\theta)$ on the interval $[-\varepsilon, +\varepsilon]$

$$a(\delta; t) = \inf \int_\varphi^{\varphi+\delta} t(\theta)\, d\theta, \qquad \varphi,\ \varphi + \delta \in [-\varepsilon, +\varepsilon], \tag{6}$$

which under our conditions is as follows:

$$a(\delta;t) = \int_{-\delta/2}^{\delta/2} t(\theta)\,d\theta = 2\int_{0}^{\delta/2} t(\theta)\,d\theta. \tag{7}$$

The inequalities (3), (4) can now be written as

$$\frac{2m}{n} \leq C_2 m\delta_m t(C\delta_m), \tag{3'}$$

$$\frac{2m}{n} \leq C_1\omega_2\left(\frac{1}{m};t\right) + \gamma_m t(C\delta_m). \tag{4'}$$

Knowing the quantity δ_m, we can find an upper estimate for m as a function of n, and consequently also an estimate for $M_n < 2m$.

IV. Let first the weight function $p(x)$ have at the point $x = -1$ a singularity of algebraic character

$$p(x) = (1+x)^\gamma p_1(x), \qquad -\frac{1}{2} < \gamma, \qquad 0 < C_3 \leq p_1(x) \leq C_4, \tag{8}$$
$$x \in [-1, -1+\eta], \qquad \eta > 0$$

(where m is taken so large that $-1 < x_1^{(m)} < x_2^{(m)} < 1+\eta$). In this case, from (7) we find the estimate $\delta_m \leq C_5 \lg m/m$, after which formulas (3), (4) give

$$\frac{2m}{n} \leq \begin{cases} C_6 \lg m\left(\dfrac{\lg m}{m}\right)^{2\gamma+1}, \\ C_1\omega_2\left(\dfrac{1}{m};t\right) + C_7\left(\dfrac{\lg m}{m}\right)^{2\gamma+1}. \end{cases} \tag{9}$$

The final determination of the estimate for M_n depends on the relative order of the quantities

$$\lg m(\lg m/m)^{2\gamma+1}, \qquad \omega_2(1/m;t), \qquad (\lg m/m)^{2\gamma+1},$$

with respect to one another; the results are collected in Table 1**.

V. If additional restrictions are imposed on the behavior of the function $p(x)$ in (8) on the entire interval $[-1,+1]$, then one can obtain the more precise estimate $\delta_m \leq C_8 \cdot 1/m$ (see (5, 6)); in particular, for this it suffices that the function $p_1(x)$ in (8) be continuous and decreasing on the interval $[-1,+1]$, with $-\frac{1}{2} \leq \gamma \leq \frac{1}{2}$ (condition C may then be discarded); using this estimate, we obtain the inequality

$$2m/n \leq C_1\omega_2(1/m;t) + C_5(1/m)^{2\gamma+1}; \tag{10}$$

if $(1/m)^{2\gamma+1} = o[\omega_2(1/m;t)]$, then we arrive at the estimate $M_n \leq \varphi^{-1}(Cn)$; if, however, $\omega_2(1/m;t) = o(1/m)^{2\gamma+1}$, then $M_n \leq C_{10} n^{1/(2\gamma+2)}$,

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* For the proof, see (2), Lemma 1.
** The function φ^{-1} is inverse to the function $\varphi(m) = m/\omega_2(1/m;t)$.

VI. Let now the weight function $p(x)$ have at the point $x = -1$ a zero of logarithmic character

$$p(x) = |\lg(1+x)|^{-\gamma} p_1(x), \qquad \gamma > 0, \qquad x \in [-1, -1+\eta]; \tag{11}$$

it is not difficult to show that in this case we have the very same estimate $\delta_m \leq C_{11} \lg m/m$; the results are collected in Table 2.

Table 1

Conditions	Estimates for M_n
$\omega_2(1/m;t) = o\left[(\lg m/m)^{2\gamma+1}\right]$	$\{n(\lg n)^{2\gamma+1}\}^{1/(2\gamma+2)}$
$(\lg m/m)^{2\gamma+1} = o[\omega_2(1/m;t)]$, $\omega_2(1/m;t) = o\left(\lg m(\lg m/m)^{2\gamma+1}\right)$	$\varphi^{-1}(Cn)$
$\lg m(\lg m/m)^{2\gamma+1} = o[\omega_2(1/m;t)]$	$n^{1/(2\gamma+2)} \lg n$

Table 2

Conditions	Estimates for M_n
$\omega_2(1/m;t) = o\left[(\lg m)^{1-\gamma}/m\right]$	$[n(\lg n)^{1-\gamma}]^{1/2}$
$(\lg m)^{1-\gamma}/m = o[\omega_2(1/m;t)]$, $\omega_2(1/m;t) = o\left((\lg m)^{2-\gamma}/m\right)$	$\varphi^{-1}(Cn)$
$(\lg m)^{2-\gamma}/m = o[\omega_2(1/m;t)]$	$[n(\lg n)^{2-\gamma}]^{1/2}$

In the case of a zero of exponential character

$$p(x) = \exp\{-1/(1+x)^{\gamma}\} p_1(x), \qquad \gamma > 0, \qquad x \in [-1, -1+\eta] \tag{12}$$

we find from (7) the estimate $\delta_m \leq C_{12} m^{-1/(2\gamma+1)}$; the final estimate for the degree of precision M_n is again as follows: $M_n \leq \varphi^{-1}(Cn)$.

VII. In conclusion we note the following: we have found an estimate for M_n by considering the behavior of the weight function $p(x)$ near the point $x = -1$; it is necessary to find an analogous estimate M_n' from consideration of its behavior near the point $x = +1$, and then take the smaller of the quantities M_n and M_n'.

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Note: Figure translations are in progress. See original paper for figures.