MIXED BOUNDARY-VALUE PROBLEMS OF GENERAL TYPE, NONSTATIONARY ON PART OF THE BOUNDARY

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Abstract

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MATHEMATICS

V. V. BARKOVSKII

MIXED BOUNDARY-VALUE PROBLEMS OF GENERAL TYPE, NONSTATIONARY ON PART OF THE BOUNDARY

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In this note we investigate certain problems for equations with partial derivatives which reduce to the Cauchy problem for an evolution equation of the type $du/dt = Au + F(t)$, where the operator A is generated by an elliptic eigenvalue problem with a spectral parameter in the boundary condition. In § 2 analogous problems are studied in which du/dt is replaced by other expressions, and in § 3 the case is considered of an operator A depending on t and $F = F(t, u)$.

1°. Let G be a bounded domain of n-dimensional space with smooth boundary $S = \Gamma_1 \cup \Gamma_2$. Consider the problem

$$\frac{\partial u(x,t)}{\partial t} = -\mathcal{L}u(x,t) + f_0(x,t), \quad x \in G,\ t > 0; \qquad B_j' u(x,t) = 0, \quad x \in \Gamma_2;$$
$$\frac{\partial (C_j u)(x,t)}{\partial t} = -B_j u(x,t) + f_j(x,t), \quad x \in \Gamma_1,\ t > 0;$$
$$u(x,0) = \psi_0(x),\ x \in G; \quad (C_j u)(x,0) = \psi_j(x),\ x \in \Gamma_1, \quad 1 \le j \le m. \tag{1}$$

Here $\mathcal{L}$ is a properly elliptic differential expression of order $2m$; $\{B_j\}_{j=1}^m$ and $\{B_j'\}_{j=1}^m$ are normal systems of differential expressions of order m_j, covering $\mathcal{L}$ (B_j' may differ from B_j by coefficients and terms whose order is less than m_j); $\{C_j\}_{j=1}^m$ is a system of differential expressions of order $l_j = 2m - m_j - 1$, completing $\{B_j\}_{j=1}^m$ to a Dirichlet system of order $2m$ (see, for example, ([1])). Problem (1) below is reduced to the Cauchy problem for an evolution equation.

We introduce the corresponding operator A. Let $C^{2m}(G \cup \Gamma_1)$ be the set of functions $2m$ times continuously differentiable in $G \cup \Gamma_1$; denote by L^2 the orthogonal sum of Hilbert spaces

$$\oplus \sum_{i=0}^{m} L_2(G^i);$$

x_0 ranges over $G^0 = G$ up to Γ_2, and $x_1 \in G^k = \Gamma_1,\ 1 \leq k \leq m$. The domain $D(A')$ of the operator A' consists of functions from $C^{2m}(G \cup \Gamma_1)$ such that $B'_j\varphi|_{\Gamma_2} = 0,\ 1 \leq j \leq m$. The operator A' acts in the Hilbert space L^2 according to the rule

$$(\varphi(x_0), (C_1\varphi)(x_1), \dots, (C_m\varphi)(x_1)) \to ((\mathcal{L}\varphi)(x_0), (B_1\varphi)(x_1), \dots, (B_m\varphi)(x_1)). \tag{2}$$

The arguments of ([2]) allow one to prove the density of $D(A_1)$ in L^2. If $\mathcal{L}$, $\{B_j\}_{j=1}^m$, and $\{B'_j\}_{j=1}^m$ are formally self-adjoint, and their coefficients and the boundary S are sufficiently smooth, then, according to Lemma 1 of ([3]), one can choose the system $\{C_j\}_{j=1}^m$ so that the operator A' will be symmetric. A' admits a closure in L^2, which we denote by A. Let A^* be the operator in L^2 adjoint to A. Using the theorems on homeomorphisms ([4]), as in ([2]), one can prove that elements of $D(A)$ and $D(A^*)$ are regular and have the form

$$U(x) = (u(x_0), (C_1u)(x_1), \dots, (C_{mu})(x_1)).$$

The regularity properties

make it possible, by integration by parts, to obtain the equality $(A^*U, V)_{L^2} = (U, A^*V)_{L^2}$ $(U, V \in D(A^*))$, from which it follows that

Theorem 1. If $\mathcal{L}$, $\{B_j\}_{j=1}^m$ and $\{B'_j\}_{j=1}^m$ are formally self-adjoint, the surface S and the coefficients of the differential expressions are sufficiently smooth, and $\min_j m_j = m$, then the operator A is self-adjoint in L^2.

This theorem generalizes Theorem 2 of ([5]).

Problem (1) in L^2 is equivalent to the Cauchy problem

$$\frac{dU(x,t)}{dt} = -AU(x,t) + F(x,t), \qquad U(x,0) = \Psi(x) \in L^2, \tag{3}$$

where $F(x,t) = (f_0(x_0,t), f_1(x_1,t), \dots, f_m(x_1,t))$, $\Psi(x) = (\psi_0(x_0), \psi_1(x_1), \dots, \psi_m(x_1))$, and $U(x,t) = (u(x_0,t), u_1(x_1,t), \dots, u_m(x_1,t))$ are regarded as functions of t with values in L^2.

Theorem 2. Suppose the hypotheses of Theorem 1 are satisfied, and $\mathcal{L}$ and $\{B_j\}_{j=1}^m$ are such that A is nonnegative; $F(x,t)$ is strongly continuous for $t \in [0, \infty)$, and

$$\sup_{0 \leq t < \infty} \|F(x,t)\|_{L^2} < \infty.$$

Then, for $t > 0$, there exists a regular solution of problem (3) satisfying the initial condition $U(\cdot, t) \to \Psi(\cdot)$ as $t \to 0$ in the sense of strong convergence in L^2. This solution is unique in L^2, depends continuously (in the sense of the L^2 norm) on $\Psi(\cdot)$, $F(\cdot, t)$, and is representable in the form $U(x) = (u(x_0, t), u_1(x_1, t), \dots, u_m(x_1, t))$, where

$$u_j(x_j, t) = \sum_{i=0}^{m} \int_{G^i} \left(\int_0^\infty \sum_{\alpha=1}^{N_\lambda} e^{-\lambda t} \varphi_{\alpha j}(x_j, \lambda) \overline{\varphi_{\alpha i}(y_i, \lambda)}\, d\sigma(\lambda) \right) \psi_i(y_i)\, dy_i +$$

$$+ \sum_{i=0}^{m} \int_0^t \int_{G^i} \left(\int_0^\infty \sum_{\alpha=1}^{N_\lambda} e^{-\lambda(t-\tau)} \varphi_{\alpha j}(x_j, \lambda) \overline{\varphi_{\alpha i}(y_i, \lambda)}\, d\sigma(\lambda) \right) f_i(y_i, \tau)\, dy_i\, d\tau, \quad (4)$$

$$0 \le j \le m.$$

Here N_λ is the dimension of the eigensubspace corresponding to the spectral point λ; $\varphi_\alpha(x, \lambda) = (\varphi_{\alpha 0}(x_0, \lambda), \varphi_{\alpha 1}(x_1, \lambda), \dots, \varphi_{\alpha m}(x_1, \lambda))$ is an eigenfunction of the operator A; $\sigma(\lambda)$ is a positive measure.

We outline the **proof**. It is not hard to see that

$$U(\cdot, t) = e^{-At}\Psi(\cdot) + \int_0^t e^{-A(t-\tau)} F(\cdot, \tau)\, d\tau$$

is a solution of problem (3). The function

$$\gamma(A) = \int_0^\infty e^{-\lambda t}\, dE_\lambda$$

(E_λ is the resolution of the identity of the operator A) is bounded for $t \ge 0$ at large λ on the spectrum of the operator A; therefore the method ([1]), transferred to the orthogonal sum of Hilbert spaces in ([7]), gives representation (4). The operators $A^N e^{-At}$ $(N = 1, 2, \dots)$ are bounded for $t > 0$; hence $U(x, t) \in D(A^N)$. The arguments of ([3]) make it possible to study the regularity properties of $D(A^N)$, from which the regularity of $U(x, t)$ follows.

Moreover, along these lines one can show that $\gamma(A)$ will be an integral operator with kernel $(K_{ij}(x_i, y_j, \lambda))_{i,j=0}^m$, and also study the singularities and regularity properties of the elements of the kernel and their connection with the eigenfunctions of the operator A.

We note that if the nonstationary boundary conditions are prescribed on the entire boundary of the domain, then the correctness of the problem considered follows from ([6]).

A theorem analogous to Theorem 2 can also be proved in the case when problem (1) is stationary inside the domain, i.e. the first equation in (1) is replaced by

$\mathcal{L}u(x,t) = 0$, while the remaining equations are retained. Similarly to §2 of ([7]), one can introduce an auxiliary self-adjoint operator in a Hilbert space

$$L^2(\Gamma_1) = \bigoplus_{i=1}^{m} \sum L_2(G^i)$$

(or its restrictions), an operator B closely connected with A. Then the problem under consideration in this Hilbert space (for definiteness, in $L^2(\Gamma_1)$) is equivalent to the Cauchy problem

$$\frac{d\hat{U}(x_1,t)}{dt} = -B\hat{U}(x_1,t) + F_1(x_1,t), \qquad \hat{U}(x_1,0) = \Psi(x_1) \in L^2(\Gamma_1);$$

$$\hat{U}(x_1,t),\ F_1(x_1,t) \in L^2(\Gamma_1).$$

There holds

Theorem 3. *If the conditions of Theorem 1 are satisfied, the nonnegative operator A has a bounded inverse, $F_1(x_1,t)$ is strongly continuous for $t \in [0,\infty)$, and*

$$\sup_{0\le t<\infty} \|F_1(x_1,t)\|_{L^2(\Gamma_1)} < \infty,$$

then problem (1), stationary in the interior of the domain, is well posed, its solution is regular and is representable analogously to (4).

2°. Problem (1) for the Schrödinger equation has, for example, the form

$$\frac{1}{i}\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) - c(x)u(x,t), \quad x \in G,\ t > 0; \qquad \frac{\partial u}{\partial n} + \sigma(x)u\Big|_{\Gamma_2} = 0;$$

$$\frac{1}{i}\frac{\partial u(x,t)}{\partial t} = -\frac{\partial u(x,t)}{\partial n}, \quad x \in \Gamma_1,\ t > 0; \qquad u(x,0) = \psi_0(x), \quad x \in G;$$

$$u(x_1,0) = \psi_1(x_1), \quad x_1 \in \Gamma_1$$

($\sigma(x)$ and $c(x)$ are nonnegative real functions; Δ is the Laplace expression; n is the exterior normal). In this case the operator A is not semibounded; however, Theorems 2 and 3 are valid with $e^{-\lambda t}$ replaced by $e^{-i\lambda t}$ and $F(x,t) \equiv 0$.

Under the assumptions of the preceding item, the hyperbolic problem

$$\frac{\partial^2 u(x,t)}{\partial t^2} = -\mathcal{L}u(x,t) + f_0(x,t), \quad x \in G,\ t > 0; \qquad B_i' u(x,t) = 0, \quad x \in \Gamma_2;$$

$$\frac{\partial^2 (C_j u)(x,t)}{\partial t^2} = -B_j u(x,t) + f_j(x,t), \quad x \in \Gamma_1,\ t > 0; \tag{5}$$

$$u(x,0) = \psi_0(x), \quad x \in G; \qquad (C_j u)(x,0) = \psi_j(x), \quad x \in \Gamma_1;$$

$$\frac{\partial u(x,0)}{\partial t} = \varphi_0(x), \quad x \in G; \qquad \frac{\partial (C_j u)(x_1,0)}{\partial t} = \varphi_j(x_1), \quad x_1 \in \Gamma_1; \qquad 1 \le j \le m$$

is equivalent to the Cauchy problem

$$\frac{d^2 U(x,t)}{dt^2} = -AU(x,t) + F(x,t), \quad t > 0;$$

$$U(x,0) = \Psi(x), \qquad \frac{\partial U(x,0)}{\partial t} = \Phi(x).$$

Theorem 4. *Let $\Psi(x) \in D(A)$, $\Phi(x) \in D(A^{1/2})$, and let $F(x,t)$ be continuously differentiable with respect to t, and for every $t \in [0,\infty)$ let $F(x,t) \in D(A^{1/2})$. Then (5) has a generalized solution (from $\widetilde{W}^{m,2}_{x,t,2}$, defined in (4)), which is unique in L^2, depends continuously (in the sense of the L^2-norm) on the initial data, and is representable in the form $U(x,t) = (u(x_0,t), u_1(x_1,t), \dots, u_m(x_1,t))$, where for each $0 \le j \le m$*

$$u_j(x_j,t) = \sum_{i=0}^{m} \int_{G^i} \left(\int_0^\infty \sum_{\alpha=1}^{N_\lambda} \cos \lambda^{1/2} t \, \varphi_{\alpha j}(x_j,\lambda) \, \overline{\varphi_{\alpha i}(y_i,\lambda)} \, d\sigma(\lambda) \right) \psi_i(y_i) \, dy_i +$$

$$+ \sum_{i=0}^{m} \int_{G^i} \left(\int_0^\infty \sum_{\alpha=1}^{N_\lambda} \lambda^{-1/2} \sin \lambda^{1/2} t \, \varphi_{\alpha j}(x_j,\lambda) \, \overline{\varphi_{\alpha i}(y_i,\lambda)} \, d\sigma(\lambda) \right) \varphi_i(y_i) \, dy_i +$$

$$+ \sum_{i=0}^{m} \int_0^t \int_{G^i} \left(\int_0^\infty \sum_{\alpha=1}^{N_\lambda} \lambda^{-1/2} \sin \lambda^{1/2} (t-\tau) \, \varphi_{\alpha j}(x_j,\lambda) \, \overline{\varphi_{\alpha i}(y_i,\lambda)} \, d\sigma(\lambda) \right) f_i(y_i,\tau) \, dy_i \, d\tau.$$

If A satisfies the conditions of Theorem 3, then the assertions of this theorem are also valid in the case when the first equation of problem (5) is

$$\mathcal{L}u(x,t) = 0.$$

In this way one can also study more general problems. For example, problems that are equivalent to the Cauchy problem

$$L(t)U(x,t) = -AU(x,t) + F(x,t), \qquad U^{(i)}(x,0) = \Psi_i(x),\ x \in G \cup \Gamma_1,$$

$$i = 0, 1, 2, \dots, k-1,$$

where $L(t)$ is a differential operator in t of order k.

3°. When the coefficients of $\mathcal{L}$ and $\{B_j\}_{j=1}^m$ depend on t, then $A = A(t)$ and $B = B(t)$. Usually, if the coefficients of the boundary expressions depend on t, then the domain of definition of the operator also depends on t, which significantly complicates the study of the Cauchy problem. In our case, the construction of $A(t)$ and $B(t)$ implies independence of t for their domains of definition.

Further, one may consider $\mathcal{L}$ and B_j containing nonlinear terms $f_i(t, u_i(x_i))$ $(0 \le i \le m)$; then the operator equation will also contain the nonlinear term

$$F(t,U) = (f_0(t, u(x_0,t)),\ f_1(t, u_1(x_1,t)), \dots,\ f_m(t, u_m(x_1,t))).$$

Therefore, in these cases as well one can obtain results analogous to ([6], [8–11]) for boundary-value problems of the type indicated above.

4°. Generalizing (5), in a number of cases one can prove the self-adjointness of the operators A and B also for unbounded domains. This makes it possible to consider problems analogous to those given above also in unbounded G. The theory of semigroups also permits the study of a broader class of problems, to which non-self-adjoint but maximally dissipative (see ([9])) operators correspond.

Kyiv Correspondence Faculty
of the Odessa Electrotechnical Institute of Communications
named after A. S. Popov

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Note: Figure translations are in progress. See original paper for figures.