NONDISCRETE TOPOLOGIZABILITY OF COUNTABLE RINGS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 519.48

MATHEMATICS

V. I. ARNAUTOV

NONDISCRETE TOPOLOGIZABILITY OF COUNTABLE RINGS

(Presented by Academician P. S. Aleksandrov on 2 IX 1969)

By a topological ring we shall mean a not necessarily associative ring in which a Hausdorff topology is defined, with the operations of the ring continuous. It is obvious that in any ring one can introduce, in a trivial way, the (discrete) topology so as to turn it into a topological ring. In finite rings this is the only possible topology. The question of the possibility of introducing a nondiscrete topology in any infinite ring has not yet been solved. In the present paper it is proved that in any countable ring one can define a nondiscrete topology.

Let R be some ring and x some variable. By a monomial in x over R we shall mean an arbitrary formal product of elements of R and x, with any placement of parentheses. The degree of a monomial is the number of occurrences of x in the given monomial. By a polynomial in x over R we shall mean an arbitrary finite sum of monomials. The degree of a polynomial is the maximal degree of the monomials occurring as summands in the polynomial. An element $a \in R$ is called a root of the polynomial $P(x)$ if $P(a) = 0$.

Analogously to the proof of Theorem 3 in (^{1}), one proves

Theorem 1. *In order that a countable ring R admit a nondiscrete topologization in which an ideal I is an open ideal, it is necessary and sufficient that, for any finite number of polynomials $P_1(x), P_2(x), \dots, P_n(x)$ over R with nonzero free terms, there exist a nonzero element of I which is not a root of any of these polynomials.*

Let M be an arbitrary set and s some natural number. By $M^{(s)}$ we denote the set of all subsets of the set M containing exactly s elements.

Lemma 1. *Let M_0 be some countable set. For any natural numbers s and r, and any partition of the set $M_0^{(s)}$ into r classes $N_1, N_2, \dots, N_r$, there exists a countable subset $M \subseteq M_0$ and a number $l \leq r$ such that $M^{(s)} \subseteq N_l$.*

Proof. We prove the assertion by induction on the number s. If $s = 1$, then the assertion of the lemma is obvious. Suppose that the lemma is true for the number s, and prove it for $s + 1$.

Let

$$M_0^{(s+1)} = \bigcup_{i=1}^{r} N_i$$

and let a_0 be some element of M_0. If we take

$$N_i' = \{A \in (M_0 \setminus \{a_0\})^{(s)} \mid \{a_0\} \cup A \in N_i\},$$

then we obtain a partition of the set $(M_0 \setminus \{a_0\})^{(s)}$ into r classes $N_1', N_2', \dots, N_r'$. By the induction hypothesis there exists an infinite subset $M_1 \subseteq M_0 \setminus \{a_0\}$ and a natural number $l_0 \leq r$ such that $M_1^{(s)} \subseteq N_{l_0}'$. Then $\{a_0\} \cup A \in N_{l_0}$ for any $A \in N_1^{(s)}$, and

$$M_1^{(s+1)} = \bigcup_{i=1}^{r} (N_i \cap M_1^{(s+1)}).$$

We now proceed with the set M_1 in the same way as we proceeded above with the set M_0. We obtain an element $a_1 \in M_1$, a natural number $l_1 \leq r$, and such a⋯

an infinite subset $M_2 \subseteq M_1 \setminus \{a_1\}$ such that $\{a_1\} \cup A \in N_{l_1}$ for any $A \subseteq M_2^{(s)}$.

Continuing this process indefinitely, we obtain sequences of elements $a_0, a_1, a_2, \dots$ from M_0 and of natural numbers $l_0, l_1, l_2, \dots$, not greater than r, and such a decreasing sequence of subsets

$$M_0 \supset M_1 \supset M_2 \supset \cdots,$$

that $a_j \in M_j$ and $\{a_j\} \cup A_j \in N_{l_j}$ for any $A_j \subseteq M_{j+1}^{(s)}$.

If we take $B_k = \{a_i \mid l_i = k\}$, then there exists a number $l \leq r$ such that B_l is an infinite set.

Take $M = B_l$ and show that M is the desired set.

Let $\{a_{i_1}, a_{i_2}, \dots, a_{i_{s+1}}\} \subseteq M^{(s+1)}$. Without loss of generality we may assume that $i_1 < i_2 < \cdots < i_{s+1}$. Since $a_{i_k} \in M_{i_k} \subseteq M_{i_1+1}$ for $k \geq 2$, it follows that $\{a_{i_2}, \dots, a_{i_{s+1}}\} \subseteq M_{i_1+1}^{(s)}$. Hence

$$\{a_{i_1}, a_{i_2}, \dots, a_{i_{s+1}}\} \subseteq \{\{a_{i_1}\} \cup A \mid A \subseteq M_{i_1+1}^{(s)}\} \subseteq N_l.$$

This proves the lemma completely.

The lemma easily implies

Theorem 2. *If M is a countable set, then for any partition of the set of all finite subsets of M into a finite number of classes $N_1, N_2, \dots, N_r$ and any natural* *number s, there exists an infinite subset $M_s \subseteq M$ and numbers $l_1, l_2, \dots, l_s$ such that $M_s^{(k)} \subseteq N_{l_k}$ for every $k \le s$.*

Theorem 3 (van der Waerden). *For any numbers m and r there exists a number $n(m, r)$ such that, under any partition of an interval of natural numbers of length $n(m, r)$ into r subsets, at least one of them contains an arithmetic progression of length m (see, for example, (2)).*

Lemma 2. *For any natural numbers m, r there exists a number $s(m, r)$ such that, under any partition of the interval of the natural series from 1 to $s(m, r)$ into r subsets, at least one of them contains numbers*

$$l_1, l_2, \dots, l_m,$$

such that any sum of the form

$$\sum_{j=1}^{q} l_{i_j},$$

where $i_j \ne i_k$, also belongs to this same subset.

Proof. Define natural numbers s_i by induction, taking $s_0 = 1$ and $s_{i+1} = n(s_i + 1, r)$. We shall show that the number $s = s_{m \cdot r}$ is the desired one.

Let the interval of the natural series from 1 to s be divided into r subsets $A_1, A_2, \dots, A_r$. For any number $0 \le j \le m \cdot r$, define a partition of the interval of the natural series from 1 to $s_{m \cdot r - j}$ into r subsets

$$A_{1,j}, A_{2,j}, \dots, A_{r,j}$$

as follows.

Take $A_{i,0} = A_i$ for $i = 1, 2, \dots, r$. Suppose that a partition of the interval of the natural series from 1 to $s_{m \cdot r - j}$, for $j < m \cdot r$, into r subsets

$$A_{1,j}, A_{2,j}, \dots, A_{r,j}$$

has already been defined. Then at least one of these subsets contains an arithmetic progression of length $1 + s_{m \cdot r - j - 1}$, i.e.

$$\{\, n_{j+1} + i p_{j+1} \mid i = 0, 1, \dots, s_{m \cdot r - j - 1} \,\}$$

is contained in some subset $A_{t,j}$. Put

$$A_{k,j+1} = \{\, i \mid i p_{j+1} \in A_{k,j} \,\}$$

for $k = 1, 2, \dots, r$. Since

$$i p_{j+1} \le n_{j+1} + i p_{j+1} \le s_{m \cdot r - j}$$

for $i \le s_{m \cdot r - j - 1}$, we have

$$\bigcup_{k=1}^{r} A_{k,j} \supseteq \{\, i p_{j+1} \mid 1 \le i \le s_{m \cdot r - j - 1} \,\}.$$

Then

$$\bigcup_{k=1}^{r} A_{k,j+1} = \{1, 2, \dots, s_{m\cdot r-j-1}\},$$

i.e. we have obtained a partition of the interval of the natural series from 1 to $s_{m\cdot r-j-1}$ into r subsets

$$A_{1,j+1}, A_{2,j+1}, \dots, A_{r,j+1}.$$

Thus, for any number $0 \le j < m \cdot r$, a partition of the interval of the natural series from 1 to $s_{m\cdot r-j}$ into r subsets

$$A_{1,j}, A_{2,j}, \dots, A_{r,j}$$

is defined, and at least one of these subsets contains an arithmetic progression

$$n_{j+1} + ip_{j+1}, \quad i = 0, 1, \dots, 1 + s_{m\cdot r-j-1}.$$

Since j takes $m \cdot r$ values, there exist a number t_0 and numbers

$$0 \le j_1 < j_2 < \dots < j_m < m \cdot r$$

such that in each of the sets A_{t_0,j_k}

$k = 1, 2, \dots, m$, contains the corresponding arithmetic progression $n_{j_{k+1}} + ip_{j_{k+1}}$, $i = 0, 1, \dots, s_{m\cdot r-j_k-1}$.

For any $1 \le k \le m$ take $l_k = (p_1 p_2 \dots p_{j_k}) n_{j_{k+1}}$ and show that the numbers $l_1, l_2, \dots, l_m$ are the desired ones.

Before completing the proof of the lemma, let us prove some properties of the sets $A_{t,j}$.

(1) If $i \in A_{t,j}$, then $(p_1 p_2 \dots p_j) i \in A_{t,0}$ for any $j \le m \cdot r$.

Indeed, from the definition of the sets $A_{t,j}$ it follows that $p_j i \in A_{t,j-1}$. Then $p_{j-1} p_j i \in A_{t,j-2}$, and so on. Applying this process j times, we obtain $(p_1 p_2 \dots p_j) i \in A_{t,0}$.

$$(2) \quad \sum_{j=1}^{i+q} n_{j+1}(p_{i+1} p_{i+2} \dots p_j) \le s_{m\cdot r-1}$$

for any i.

Indeed, if $q = 0$, then $n_{i+1} \le s_{m\cdot r-i}$. Suppose that

$$\sum_{j=i}^{i+q} n_{j+1}(p_{i+1} p_{i+2} \dots p_j) \le s_{m\cdot r-i}$$

for any i. Then

$$\sum_{j=1}^{i+q+1} n_{j+1}(p_{i+1}p_{i+2}\dots p_j) = n_{i+1} + p_{i+1}\left[\sum_{j=i+1}^{i+1+q} n_{j+1}(p_{i+2}p_{i+3}\dots p_j)\right] \le$$

$$\le n_{i+1} + p_{i+1}s_{m\cdot r-i-1} \le s_{m\cdot r-i}.$$

Let us now proceed to complete the proof of the lemma. Since, by property (2),

$$\sum_{i=2}^{q}(p_{j_{k_1}+2}p_{j_{k_1}+3}\dots p_{j_{k_i}})\, n_{j_{k_i}+1} \le$$

$$\le \sum_{i=j_{k_1}+1}^{j_{k_q}} n_{i+1}(p_{j_{k_1}+2}p_{j_{k_1}+3}\dots p_i) \le s_{m\cdot r-j_{k_1}-1},$$

we have

$$n_{j_{k_1}+1} + p_{j_{k_1}+1}\left[\sum_{i=2}^{q}(p_{j_{k_1}+2}p_{j_{k_1}+3}\dots p_{j_{k_i}})\, n_{j_{k_i}+1}\right] \in A_{t_0,j_{k_1}}.$$

Then, by property (1), we have

$$\sum_{i=1}^{q} l_{k_i} = (p_1p_2\dots p_{j_{k_1}})\left[n_{j_{k_1}+1} + \sum_{i=2}^{q}(p_{j_{k_1}+1}\dots p_{j_{k_i}})n_{j_{k_i}+1}\right] \in A_{t_0,0}.$$

This proves the lemma completely.

Lemma 3. If $P(x)$ is a polynomial of degree m without constant term over a ring R, then for any element $a \in R$

$$P(x+a) = P(x) + P(a) + Q(x),$$

where $Q(x)$ is a polynomial of degree at most $m-1$ without constant term.

The proof is obvious.

Lemma 4. If for a polynomial $P(x)$ of degree m over a ring R there exist elements $a_1, a_2, \dots, a_{m+1}$ such that every sum of the form $\sum_{i=1}^{q} a_{k_i}$, where $k_i \neq k_j$, is a root of $P(x)$, then the constant term of the polynomial $P(x)$ is equal to zero.

Proof. The proof is easily carried out by induction on the degree of the polynomial $P(x)$.

Theorem 4. For any finite number of polynomials over a ring R with nonzero constant terms, in any infinite subgroup I of the addiof the additive group of the ring R there exists a nonzero element a which is not a root of any of these polynomials.

Proof. Suppose the contrary, i.e., that there is a finite number of polynomials $P_1(x), P_2(x), \ldots, P_r(x)$ over R with nonzero constant terms such that every nonzero element of I is a root of one of these polynomials. Choose a countable set M of elements of I such that any finite sum of pairwise distinct elements $a_i \in M$ is nonzero.

We now define a partition of the set of all finite subsets of the set M into r classes $N_1, N_2, \ldots, N_r$ as follows: assign the subset $\{c_1, c_2, \ldots, c_n\}$ of the set M to the class N_j, where

$$j = \min\left\{k \mid P_k\left(\sum_{i=1}^{n} c_i\right) = 0\right\}.$$

If m is the greatest degree of the polynomials $P_1(x), P_2(x), \ldots, P_r(x)$ and $s = s(m+1, r)$, then by Theorem 2 there exists an infinite subset M_s such that $M_s^{(k)} \subseteq N_{l_k}$ for any $k \leq s$.

We now define a partition of the interval of the natural numbers from 1 to s into r subsets $A_1, A_2, \ldots, A_r$ as follows: put the number $i \leq s$ into the subset A_j if and only if $M_s^{(i)} \subseteq N_j$.

By Lemma 2 there exist natural numbers $j_0 \leq r$ and $l_1 < l_2 < \cdots < l_{m+1} < s$ such that any sum $l_{i_1} + l_{i_2} + \cdots + l_{i_q}$, where $i_j \neq i_k$, belongs to the subset A_{j_0}. Then

$$M_s^{\left(\sum_{j=1}^{q} l_{i_j}\right)} \subseteq N_{j_0}.$$

If $M_s = \{b_1, b_2, \ldots\}$, then define the elements $a_1 = b_1 + b_2 + \cdots + b_{l_1}$,

$$a_2 = b_{l_1+1} + b_{l_1+2} + \cdots + b_{l_1+l_2}, \ldots, \quad a_{m+1} = b_{l_1+l_2+\cdots+l_m+1} + b_{l_1+l_2+\cdots+l_m+2} + \cdots$$

$$\cdots + b_{l_1+l_2+\cdots+l_m+l_{m+1}}.$$

From the definition of the classes N_j it follows that any sum $\sum_{j=1}^{q} a_{i_j}$ is a root of the polynomial $P_{j_0}(x)$. Since the degree of the polynomial $P_{j_0}(x)$ is no greater than m, by Lemma 4 the constant term of $P_{j_0}(x)$ is equal to zero. We have obtained a contradiction with the condition of the theorem. Consequently, in I there is a nonzero element which is not a root of any of the polynomials $P_j(x)$.

From Theorems 1 and 4 it follows that

Theorem 5. For any infinite ideal I of a countable ring R, there exists a nondiscrete topologization of the ring R in which I is an open ideal.

Corollary. Every countable ring admits a nondiscrete topologization.

Institute of Mathematics with Computing Center
Academy of Sciences of the MSSR
Kishinev

Received
22 VIII 1969

References

1. V. I. Arnautov, *Sibirsk. Mat. Zh.*, **9**, No. 6 (1968).

2. A. Ya. Khinchin, *Three Pearls of Number Theory*, Moscow, 1948.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.