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Abstract

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MATHEMATICS

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ON THE GREEN'S FUNCTION OF SOME SINGULAR ELLIPTIC OPERATORS

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Let R_{n+1} be the $(n+1)$-dimensional Euclidean space of points (x, y), where $x = (x_1, \dots, x_n)$, $x_{n+1} = y$. Denote by R^+_{n+1} the open half-space of points $\{(x, y), y > 0\}$. Let Ω^+ be a domain situated in R^+_{n+1} and adjoining the hyperplane $y = 0$. Put

$$D^\alpha = D^\alpha_x = i^{|\alpha|}\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$

$$B_y = i^2\left(\frac{\partial^2}{\partial y^2} + \frac{2\nu+1}{y}\frac{\partial}{\partial y}\right), \qquad 2\nu+1,\ y > 0.$$

Let $L(x, y; D_x, B_y)$ be a linear differential operator of the form

$$L = \sum_{|\alpha|+2\beta \le 2m} a_{\alpha,\beta}(x, y) D^\alpha_x B^\beta_y, \tag{1}$$

where the coefficients $a_{\alpha,\beta}(x, y)$ are assumed to be infinitely differentiable in the domain Ω^+, and the operator L is B-elliptic, i.e. the polynomial

$$L_0(x, y; \xi, \eta^2) = \sum_{|\alpha|+2\beta = 2m} a_{\alpha,\beta}(x, y)\xi^\alpha \eta^{2\beta} \tag{2}$$

is positive definite with respect to (ξ, η) for all $y \ge 0$ from the domain Ω^+.

Let Ω^+_1 be a subdomain of the domain Ω^+, lying entirely in it and adjoining the hyperplane $y = 0$.

On the set $C_0^\infty(\Omega^+_1)$ put

$$(f,g)_\nu^{(m)} = \int_{\Omega_1^+} \sum_{|\alpha|+2\beta\le m} D_x^\alpha B_y^\beta f\, D_x^\alpha B_y^\beta g\, y^{2\nu+1}\,dx\,dy, \tag{3}$$

$$(f,g)_\nu = \int_{\Omega_1^+} fg\, y^{2\nu+1}\,dx\,dy,$$

$$\|f\|^2_{H_\nu^m} = (f,f)_\nu^{(m)}, \qquad \|f\|^2_{L_{2,\nu}} = (f,f)_\nu. \tag{4}$$

Completing C_0^∞ with respect to the indicated norm, we obtain the Hilbert space of functions $\mathring{H}_\nu^m$. Let $f,g\in C_0^\infty(\Omega_1^+)$.

Integrating by parts, we obtain that

$$(Lf,g)_\nu = L(f,g) = \int_{\Omega_1^+} \sum a_{\alpha',\beta'}^{(\alpha,\beta)}(x,y) D_x^\alpha B_y^\beta f\, D_x^{\alpha'} B_y^{\beta'} g\, y^{2\nu+1}\,dx\,dy. \tag{5}$$

Obviously, the indicated form is bounded from above in $\dot{H}_\nu^m$. From the proof of work ([1]) it follows that the form $L(f,g)$ is bounded from below in the following sense.

Let t be a sufficiently large positive number. Put

$$L_t(f,g) = L(f,g) + t(f,g)_v,$$

$$(f,g)_{v,t} = (f,g)_v^{(m)} + t(f,g)_v.$$

Then there exist a positive number t_0 and $c>0$ such that

$$c^{-1}(f,f)_{v,t} \le |L_t(f,f)| \le c(f,f)_{v,t} \qquad (t>t_0) \tag{6}$$

for all $f\in \mathring{H}_v^m$.

Therefore the relation

$$L_t(f,f') = (N_t f, f')_{v,t}, \qquad f, f', N_t f \in \mathring{H}_v^m$$

defines a bounded linear operator N_t acting from $\mathring{H}_v^m$ into $\mathring{H}_v^m$, and this operator has a bounded inverse.

Let, as before, $L_{2,v}(\Omega_1^+)$ be the set of functions square summable with weight y^{2v+1}.

The equation

$$(f, f')_v = (M_t f, f')_{v,t},$$

where $f \in L_{2,v}$, $M_t f, f' \in \overset{\circ}{H}{}^m_v$, defines a completely continuous operator acting from $L_{2,v}$ into $\overset{\circ}{H}{}^m_v$. Therefore

$$(f, f')_v = L_t(G_t f, f'),$$

where $f \in L_{2,v}$, $f' \in \overset{\circ}{H}{}^m_v$, $G_t = N_t^{-1} M_t$, defines a completely continuous linear operator G_t acting from $L_{2,v}$ into $\overset{\circ}{H}{}^m_v$, which we shall call the Green function corresponding to the differential operator $L_t = L + t$. The operator G_t maps $L_{2,v}$ into $\overset{\circ}{H}{}^m_v$, and the operator G_t^{-1} is an extension of the differential operator L_t, whose graph is the set of all pairs $\{f, L_t f\}$, where $f \in \overset{\circ}{H}{}^m_v$, $2m$ times continuously differentiable (with respect to y, application of the operator B_y the corresponding number of times is allowed) and, moreover, $L_t f \in L_{2,v}$.

Indeed, if $h \in C_0^\infty$, then

$$L_t(G_{tL} t f, h) = (L_t f, h)_v = L_t(f, h),$$

in other words $G_{tL} t f = f$.

To obtain an estimate of the Green function we proceed as follows. For sufficiently large t we have

$$(f, f')_v = L_t(G_t f, f'), \tag{7}$$

where $f \in L_{2,v}$, $G_t f, f' \in \overset{\circ}{H}{}^m_v$.

Since

$$\|f\|^2_{v,t} = (f, f)^{(m)}_v + t(f, f)_v \geq t\|f\|^2_{L_{2,v}},$$

then, taking (6) into account, we have

$$tc^{-1}\|G_t f\|^2_{L_{2,v}} \leq c^{-1}\|G_t f\|^2_{v,t} \leq |L_t(G_t f, G_t f)| \leq \|f\|_{L_{2,v}} \|G_t f\|_{L_{2,v}}. \tag{8}$$

Therefore the estimate is valid

$$\|G_t\|_{L_{2,v}} \leq \frac{c}{t}. \tag{9}$$

It was shown above that if $f \in C_0^\infty(\Omega_1^+)$, then

$$G_{tL} t f = f. \tag{10}$$

Let L_t^* be the operator adjoint to L_t:

$$(L_t f, f')_v = (f, L_t^* f')_v,$$

where $f, f' \in C_0^\infty$. If $f \in \overset{\circ}{H}{}_v^m$ and $f' \in C_0^\infty$, then

$$(G_t f, L_t^* f')_v = L_t(G_t f, f') = (f, f')_v, \tag{11}$$

Let k be a positive integer. Set

$$C(f, f') = (G_t^k f, f')_\nu t^k$$

and, moreover, set

$$b = \frac{L_t^k}{t^k}.$$

The bilinear form $C(f, f')$ is bounded, i.e.

$$|C(f, f')| \leq c \|f\|_{L_{2,\nu}} \|f'\|_{L_{2,\nu}},$$

and, according to relation (7),

$$C(bf, f') = (G_t^k L_t^k f, f')_\nu = (f, f')_\nu,$$

where the functions $f, f' \in C_0^\infty(\Omega_1^+)$.

If b^* is the operator adjoint to b, then from (11) it follows that

$$C(f, b^* f') = (G_t^k f, L_t^* f')_\nu = (f, f')_\nu$$

for all $f, f' \in C_0^\infty$. Putting $\tau^{2m} = t$, we obtain

$$b = b(\tau; x, y; D_x, B_y) = \sum b_{\alpha,\beta}(\tau; x, y) \tau^{-|\alpha| - 2\beta} D_x^\alpha B_y^\beta.$$

Using the properties of the polynomial $b(\tau; x, y; \xi, \eta^2)$, one can show that G_t^k has a continuous kernel $g_t^{(k)}(x, y; u, v)$ with the following property:

$$\lim_{t \to \infty} t^{k-\gamma} g_t^{(k)}(x, y; u, v) = \delta_{x,y}^{(u,v)} C_\nu^{(n)} \int_{R_{n+1}^+} \frac{[j_\nu(t\eta)]^2 \eta^{2\nu+1}}{|L_0(x, y; \xi, \eta^2) + 1|^k} \, d\xi \, d\eta, \tag{12}$$

if $2mk > n + 1 + 2\nu + 1$, where

$$\gamma = \frac{n + 1 + 2\nu + 1}{2m}.$$

Here $j_\nu(s)$ is the Bessel function. The symbol $\delta_{x,y}^{(u,v)}$ is equal to 1 if $(x,y)=(u,v)$, and is equal to 0 in the remaining cases. For $(x,y)=(u,v)$ the limiting equality (12) remains valid up to the hyperplane $y=0$. In this case the presence of $2\nu+1$ in the exponent of t is essential. Formula (12) takes the form

$$\lim_{t\to\infty} t^{k-\gamma} g_t^{(k)}(x,0;x,0) = C_\nu^{(n)} \int_{R_{n+1}^+} \frac{\eta^{2\nu+1}}{\left[L_0(x,0;\xi,\eta^2)+1\right]^k}\, d\xi\, d\eta, \qquad (13)$$

where

$$\gamma = \frac{n+1+2\nu+1}{2m}, \qquad 2km > n+1+2\nu+1.$$

If L_t is self-adjoint, then by virtue of property (6) the form $(G_t f, f)_\nu \geq 0$ for sufficiently large t, so that G_t is a self-adjoint positive operator. But then there exists a complete system $\varphi_1, \varphi_2, ...$ of eigenfunctions of the operator G_t (t fixed) with positive eigenvalues

$$(\lambda_1+t)^{-1} \geq (\lambda_2+t)^{-1} \geq \cdots.$$

By the properties of the operator G_t, the necessary and sufficient condition that $\varphi \in L_{2,\nu}$ satisfy the equation

$$G_t\varphi = (\lambda+t)^{-1}\varphi$$

is that $\varphi \in \dot{H}_\nu^m$ and satisfy the relation

$$(\lambda+t)(\varphi,f)_\nu = L_t(\varphi,f)$$

for all $f \in \dot{H}_\nu^m$. Therefore $G_t\varphi = (\lambda+t)^{-1}\varphi$ means that $G_s\varphi = (\lambda+s)^{-1}\varphi$, and conversely, if t, s are sufficiently large. Consequently,

$$G_t\varphi_j = (\lambda_j+t)^{-1}\varphi_j$$

for all t. It turns out that, for $2mk > n+1+2\nu+1$,

$$g_t^{(k)}(x,y;u,v) = \sum_j (\lambda_j+t)^{-k}\varphi_j(x,y)\varphi_j(u,v) \qquad (14)$$

and that the equality

$$\int_{\Omega_1^+} g_t^{(k)}(x,y;u,v) y^{2\nu+1}\, dx\, dy = \sum_j (\lambda_j+t)^{-k} = \operatorname{tr} G_t^k \qquad (15)$$

holds.

From relations (12)–(15), by applying Tauberian theorems, it is not difficult to obtain the asymptotic distribution of the eigenvalues and eigenfunctions for the singular elliptic operator indicated at the beginning. Indeed, putting, for example,

$$\sigma(\lambda) = \sum_{\lambda_j \le \lambda} |\varphi_j(x, y)|^2,$$

by a Tauberian theorem from (12) we find that

$$\sigma(\lambda) \sim C_\nu^{(n)} \frac{2m}{\pi(n+1+2\nu+1)} \sin\frac{n+1+2\nu+1}{2m}\pi\lambda^{\frac{n+1+2\nu+1}{2m}} \times \\ \times \int_{R_{n+1}^+} \frac{[j_\nu(\lambda\eta)]^2\eta^{2\nu+1}}{L_0(x, y; \xi, \eta^2)+1}\, d\xi\, d\eta. \tag{16}$$

Let us note that if the point (x, y) lies inside the domain Ω_1^+, then the operator L becomes an ordinary regular elliptic operator, and in this case our results coincide with the corresponding results of L. Gårding ([2]).

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References

[1] I. A. Kipriyanov, DAN, 181, No. 4 (1968).
[2] L. Gårding, Math. Scand., 1, 237 (1953).

Note: Figure translations are in progress. See original paper for figures.