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Abstract

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HYDROMECHANICS

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THE PROBLEM OF GENERATION OF A MAGNETIC FIELD IN THE PRESENCE OF ACOUSTIC TURBULENCE

(Presented by Academician R. Z. Sagdeev on 3 III 1970)

One of the basic problems of magnetohydrodynamic turbulence consists in clarifying the question of whether a turbulent medium is stable with respect to weak perturbations of the magnetic field, i.e., whether weak magnetic fields will grow with time. The main difficulty is the absence of a small parameter in ordinary hydrodynamic turbulence ([1]). Such a small parameter can be found in acoustic turbulence, namely $\tau v/\lambda$, where τ is the correlation time at one point, λ is the characteristic wavelength, and v is the characteristic velocity.

In problems of generation of the magnetic field $\mathbf{H}$, the velocity field $\mathbf{v}$ is assumed to be prescribed, while $\mathbf{H}$ satisfies the equation

$$\partial \mathbf{H}/\partial t = \mathrm{rot}[\mathbf{v}, \mathbf{H}] + v_m \Delta \mathbf{H}, \tag{1}$$

where v_m is the magnetic viscosity.

I. We shall assume the velocity field to be stationary and distributed homogeneously and isotropically in space. It is convenient to pass to $k-\omega$-space

$$\mathbf{v} = \int \frac{\mathbf{k}}{k}\, \varphi(\mathbf{k}, \omega) e^{i[(\mathbf{kr})-\omega t]}\, d\mathbf{k}\, d\omega.$$

Next, averaging over the ensemble, we obtain

$$\langle u_i(\mathbf{k}, \omega) u_j^*(\mathbf{k}', \omega')\rangle = \frac{k_i k_j}{k^2}\, I(k, \omega)\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega'); \tag{2}$$

$$\langle v_i(\mathbf{x}, t) v_j(\mathbf{x}+\mathbf{r}, t+s)\rangle = \int \frac{k_i k_j}{k^2}\, f(k, s) e^{i(\mathbf{kr})}\, d\mathbf{k};$$

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$$f(k, s) = \int I(k, \omega) e^{-i\omega s}\, d\omega. \quad (3)$$

Let $E(k)$ be the power spectrum of the acoustic oscillations; then $E(k) = 4\pi k^2 f(k, 0)$. The correlation time may be introduced in the following way:

$$\tau = \int dk \int_0^\infty ds\, f(k, s) \Big/ \int dk\, f(k, 0) = \pi \int dk\, I(k, 0) \Big/ \int dk\, f(k, 0). \quad (4)$$

We now determine $I(k, 0)$. For this purpose we represent the density ρ and the velocity as follows: $\rho = \rho_0 + \rho_1 + \rho_2$; $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ ($\rho_0 \gg \rho_1 \gg \rho_2$, $\mathbf{v}_1 \gg \mathbf{v}_2$). The Fourier images in $k-\omega$-space of the quantities $\rho_1, \rho_2, \mathbf{v}_1, \mathbf{v}_2$ will be denoted by $\rho_1(\mathbf{k}, \omega), \rho_2(\mathbf{k}, \omega), \frac{\mathbf{k}}{k}\varphi_1(\mathbf{k}, \omega), \frac{\mathbf{k}}{k}\varphi_2(\mathbf{k}, \omega)$. Linearization of the equations of gas dynamics gives, as is known, a solution in the form of acoustic waves, with

$$\rho_1(\mathbf{k}, \omega) = \frac{1}{c}\rho_0 \varphi_1(\mathbf{k}, \omega);$$

$$\langle \varphi_1(\mathbf{k}, \omega) \varphi_1^*(\mathbf{k}', \omega') \rangle = \delta(\mathbf{k} - \mathbf{k}')\delta(\omega - \omega') f(k, 0) \frac{1}{2}[\delta(\omega - ck) + \delta(\omega + ck)] \quad (5)$$

(c is the speed of sound). To obtain $I(k, 0)$ we write the equation for the correction of second approximation (it is sufficient to use only the equation–

continuity equation):

$$\rho_0 k \varphi_2(\mathbf{k}, \omega) = \omega \rho_2(\mathbf{k}, \omega) - \int \rho_1(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1) \frac{(\mathbf{k}\mathbf{k}_1)}{k_1} \varphi_1(\mathbf{k}_1, \omega_1)\, d\mathbf{k}_1\, d\omega_1. \quad (6)$$

Multiplying (6) by $\varphi_2^*(\mathbf{k}', \omega')$, using (5), and averaging, the expression on the right-hand side of (6), when multiplied by the complex conjugate, then gives a fourth-order moment. Here one may use the random-phase approximation and integrate over $d\mathbf{k}'$ and $d\omega'$; then we obtain ($\mathbf{p} = \mathbf{k} - \mathbf{q}$):

$$I(\mathbf{k}, 0) = \frac{1}{2c^2k^2} \int \left[\frac{(\mathbf{k}\mathbf{q})^2}{q^2} + \frac{(\mathbf{k}\mathbf{q})(\mathbf{k}\mathbf{p})}{qp} \right] f(p, 0) f(q, 0) \delta(cp - cq)\, d\mathbf{q} =$$

$$= \frac{\pi k}{2c^3} \int_{k/2}^\infty f^2(q, 0)\, dq;$$

$$\pi = \frac{\pi^2}{16c^3} \int k^2 f^2\left(\frac{k}{2}, 0\right) dk \Big/ \int f(k, 0)\, dk. \quad (7)$$

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In order of magnitude $\tau \approx (v^3/c^3)\lambda/v$, so that $\tau \ll \lambda/v$.

II. To derive the equation for the spectral function of the magnetic field $B(k,t)$, we shall use the Fourier representation (1) in k-space

$$\mathbf{H}(\mathbf{k},t) = \mathbf{H}(\mathbf{k},0)e^{-k^2\nu_m t} + i\int_0^t dt_1 e^{-k^2\nu_m(t-t_1)}\int d\mathbf{q}\left[\mathbf{k}\left[\frac{\mathbf{p}}{p}\psi(\mathbf{p},t_1)\mathbf{H}(\mathbf{q},t_1)\right]\right]. \tag{8}$$

$\mathbf{H}(\mathbf{k},t)$ and $\psi(\mathbf{k},t)\mathbf{k}/k$ are the Fourier transforms of $\mathbf{H}(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)$.

We use a perturbation-theory expansion in the velocity:

$$\mathbf{H}(\mathbf{k},t) = \sum_{n=0}^{\infty}\mathbf{H}^{(n)}; \qquad \mathbf{H}^{(0)} = \mathbf{H}(\mathbf{k},0)e^{-\nu_m k^2 t},$$

$$\mathbf{H}^{(n)} = i\int_0^t dt_1 e^{-\nu_m k^2(t-t_1)}\int d\mathbf{q}\left[\mathbf{k}\left[\frac{\mathbf{p}}{p}\psi(\mathbf{p},t_1)\mathbf{H}^{(n-1)}(\mathbf{q},t_1)\right]\right]. \tag{9}$$

We restrict ourselves to terms of second order in ψ (which is justified when $\tau \ll \lambda/v$). Multiplying (9) by $H_j^*(\mathbf{k}',t)$, we shall assume: 1) $\mathbf{H}(\mathbf{k},0)$ is statistically independent of $\psi(\mathbf{k},t)$; 2) ν_m is small, so that in (9) one may put $\exp(-\nu_m k^2 t) = 1-\nu_m k^2 t$; 3) $\langle H_i(\mathbf{k},0)H_j^*(\mathbf{k}',0)\rangle = B(k,0)\delta(\mathbf{k}-\mathbf{k}')\times(\delta_{ij}-k_ik_j/k^2)$.

Averaging the resulting expression and integrating over $d\mathbf{k}'$, we obtain

$$B(k,t) = B(k,0) - 2B(k,0)\left[\int_0^t ds\,(t-s)\int f(q,s)\frac{(\mathbf{kq})^2}{q^2}\,d\mathbf{q} + \nu_m k^2 t\right] +$$

$$+\int_0^t ds\,(t-s)\int f(p,s)B(q,0)\frac{(\mathbf{kp})^2q^2 + k^2(\mathbf{qp})^2}{p^2q^2}\,d\mathbf{p}. \tag{10}$$

From the asymptotic behavior of (10) at large t ($t \gg \tau$), one can construct an equation for $B(k,t)$:

$$\frac{\partial B(k,t)}{\partial t} + 2(\chi+\nu_m)k^2B(k,t) = \pi\int I(p,0)B(q,t)\frac{(\mathbf{kp})^2q^2 + k^2(\mathbf{qp})^2}{p^2q^2}\,d\mathbf{p}; \tag{11}$$

$$\chi = \frac{\pi}{3}\int I(q,0)\,d\mathbf{q} = \frac{\pi^2}{48c^3}\int k^2 f^2\left(\frac{k}{2},0\right)\,d\mathbf{k}.$$

(11) can be transformed to r-space:

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$$\frac{\partial B_1}{\partial t} = 2\left\{AB_1'' + \frac{A_1}{r}B_1' - 2\frac{v'}{r}B_1\right\};$$

$$B_1 = r^{-3}\int_0^r \rho^2 B(\rho,t)\,d\rho; \qquad B(r,t) = \int B(k,t)e^{i(\mathbf{k},\mathbf{r})}\,d\mathbf{k};$$

$$v(r) = \pi\int I(k,0)e^{i(\mathbf{k},\mathbf{r})}\,d\mathbf{k}; \qquad A = \chi + \nu_m - (rv_1' + v_1); \tag{12}$$

$$A_1 = 4(\chi + \nu_m) - (\nu' r + 2\nu_1' r + 4\nu_1); \qquad \nu_1 = r^{-3}\int_0^r \nu(\rho)\rho^2\,d\rho.$$

III. We shall seek the eigenfunctions (12) by the substitution $B_1 = he^{-2Et}$; then, by the substitution

$$z = h_1 \exp\left[\frac{1}{2}\int_0^r \frac{d\rho\,A_1(\rho)}{\rho A(\rho)}\right]$$

we pass to the Schrödinger equation

$$z'' + m(E-U)z = 0, \tag{13}$$

where the "mass" is $m = 1/A$, and the potential is

$$U = 2\nu'/r + \frac{1}{2}A(A_1/rA)' + A_1^2/4r^2A.$$

As $r \to 0$, $U \to 2\nu_m/r^2$; as $r \to \infty$, $U \to 2(\chi+\nu_m)/r^2$. The wave function of a bound state in the potential corresponds to an unstable (growing) solution in the present problem, the eigenenergy of the state being the growth increment with the opposite sign. A bound state appears if U has a potential well. The well arises if $\chi \gg \nu_m$; indeed, in this case, for r such that

$$\nu(0)/\nu_2(0) \gg r^2 \gg \nu_m/\nu_2(0), \qquad \text{where} \quad \nu_2(r) = \int \pi I(q,0)q^2\,dq,$$

$$U \simeq \nu_2(0)/15.$$

The presence of a potential well is not a sufficient condition for the existence of eigenfunctions with $E < 0$—the well may prove insufficiently deep; moreover, the eigenfunction must have the properties of a correlation function (its Fourier image $B(k,t)$ must be positive). To determine a sufficient condition, let us turn to equation (11). By the substitution $B = he^{-2Et}$ we pass to the eigenvalue

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problem for an integral equation. To find the minimal E, we formulate the variational principle

$$E = \left[(\chi + \nu_m) \int k^2 h^2 \, dk - \frac{\pi}{2} \int dk \, dp \, h(k) h(q) I(p,0) \frac{(\mathbf{kp})^2 q^2 + k^2 (\mathbf{qp})^2}{q^2 p^2} \right] \Big/ \int h^2 \, dk; \tag{14}$$

$$\delta E = 0.$$

Assign to $h(k)$ the following form: for $k \le k_1$,

$$h(k) = k_0^{-1/2} k_1^{-3} k^2 \exp(-k_1/2k_0),$$

where

$$\nu_2(0)/\nu(0) \ll k_1^2 \ll \nu_2(0)/\nu_m, \qquad k_0^2 \simeq \nu_2(0)/\nu_m;$$

for $k \ge k_1$,

$$h(k) = k^{-1} k^{-1/2} \exp(-k/2k_0).$$

For $k^2 < \nu_2(0)/\nu(0)$ only a negligible part of the energy $h(k)$ is concentrated; therefore the right-hand side of (14) can be simplified:

$$E = \left[\nu_m \int k^2 h^2 \, dk - \nu_2(0)/5 \int \left\{ \frac{2}{3} h^2 + 2hh'k + h''hk^2 \right\} dk \right] \Big/ \int h^2 \, dk =$$

$$= -\frac{7}{30} \nu_2(0) + 2k_0^2 \nu_m + \nu_2 \alpha(k_1/k_0),$$

where $\alpha(k_1/k_0)$ is a small quantity of order k_1/k_0. It is clear that if $k_0^2 \lesssim 7\nu_2(0)/60\nu_m$, then our $h(k)$ gives the functional (14) a negative value. Consequently, the eigenfunctions of problem (11) with $E < 0$ exist. Thus, acoustic turbulence is unstable with respect to perturbations of the magnetic field. It is not difficult to show that $\min U(r) = -\frac{2}{3}\nu_2(0)$; therefore the growth increment of the field is

$$\gamma \simeq \nu_2(0) = \frac{\pi}{24c^3} \int k^4 f^2 \left(\frac{k}{2}, 0 \right) dk. \tag{15}$$

We see that the most essential quantity determining the possibility of field generation is

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$$\nu(0)/\nu_m = 3\chi/\nu_m \simeq (v^3/c^3)v\lambda/\nu_m = S = \mathrm{M}^3\mathrm{Re}_m, \tag{16}$$

(M is the Mach number, Re_m is the magnetic Reynolds number). For instability it is necessary that $S \gg 1$.

IV. The velocity field of acoustic turbulence is potential. The vortical component is generated in the same way as the magnetic one, since the equations for $\mathbf{H}$ and $\mathrm{rot}\,\mathbf{v}$ are analogous and $\mathrm{rot}\,\mathbf{v}(\mathbf{r}, 0)$ may be taken to be statistically independent of the potential component. Here the quantity most essential for generation is $S_\omega = \mathrm{M}^3\mathrm{Re}$. If $S_\omega \gg 1$, the generation of vortices is analogous to the instability described above; if $S_\omega \ll 1$, there occurs growth of the vortical component v_ω as a nonlinear effect—a phenomenon known as acoustic streaming ([2]).

The growth of v_ω for $S_\omega \gg 1$ is limited by turbulent viscosity: the point is that the vortex-vortex interaction has the usual character of energy transfer into the region of large k. For a rough estimate of the energy of the established level v_ω, we note that, since the width of the potential pit is $\sim \lambda$, the energy v_ω^2 is concentrated at $k \sim 1/\lambda$. Let us write the equation for v_ω:

$$\frac{d}{dt}v_\omega^2 = \gamma v_\omega^2 - v_\omega^2/\lambda.$$

In the stationary state $v_\omega \sim \lambda\gamma$. Let us calculate v_ω for the following form of E: for $k < 1/\lambda$, $E = 0$; for $1/\lambda \leq k \leq k_\nu$, $E = v^2\lambda^{1-\alpha}k^{-\alpha}$, where k_ν is the spectral cutoff due to viscosity; for $k > k_\nu$, $E = 0$. Here there are 3 cases: 1) $\alpha > 3/2$, then $\gamma \sim \mathrm{M}^3 v/\lambda$ and $v_\omega \sim \mathrm{M}^3 v$; 2) $1 < \alpha < 3/2$, then $\gamma \sim \mathrm{M}^3(\lambda k_\nu)^{2-2\alpha}vk_\nu$, $k_\nu = \mathrm{Re}^{2/(1+\alpha)}\lambda^{-1}$, $v_\omega \sim v\mathrm{M}^3\mathrm{Re}^{2(3-2\alpha)/(1+\alpha)} \leq vS_\omega$ (for $\alpha < 3/2$, $v_\omega \sim \mathrm{M}^3 v$); 3) $\alpha = 3/2$, then $\gamma \sim \mathrm{M}^3\ln(k_\nu\lambda)v/\lambda$, $v_\omega \sim \mathrm{M}^3 v \ln \mathrm{Re}$.

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Note: Figure translations are in progress. See original paper for figures.

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