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Abstract

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MATHEMATICS

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ON A FREQUENCY SOLUTION OF ALGORITHMICALLY UNSOLVABLE MASS PROBLEMS

(Presented by Academician P. S. Novikov on 4 IX 1969)

1°. Suppose a certain mass problem $\mathcal{M}$ with a fixed numbering of individual problems is given. If $\mathcal{M}$ is an algorithmically unsolvable problem, then it is impossible to solve all these problems by a single method. But the question arises: perhaps the "majority" of the individual problems can nevertheless be solved by a single method?* We shall say that the mass problem $\mathcal{M}$ is algorithmically solvable with frequency $1-\varepsilon$ on infinitely many initial segments if there exists an algorithm Ω such that, for infinitely many natural n,

$$[\mathcal{M}, \Omega, n]/n \geqslant 1-\varepsilon,$$

where $[\mathcal{M}, \Omega, n]$ is the number of those individual problems among the first n which the algorithm Ω solves correctly; here it is required that the algorithm Ω be applicable to every individual problem from $\mathcal{M}$ (i.e., halt when solving it). The main purpose of this article is to show that a number of typical unsolvable mass problems can be algorithmically solved with frequency $1-\varepsilon$ on infinitely many initial segments (ε is an arbitrary preassigned number greater than 0).

2°. As a typical example of mass problems let us consider the problem of membership in a set M, where M is an arbitrary recursively enumerable set. In this case the notion of solvability with frequency $1-\varepsilon$ can be made precise as follows: the problem of membership in the set M is algorithmically solvable with frequency $1-\varepsilon$ on infinitely many initial segments if there exists a general recursive predicate $P(x)$ such that, for infinitely many natural n,

$$[M, P, n]/n \geqslant 1-\varepsilon,$$

where $[M, P, n]$ is the number of those natural x, not exceeding n, for which $P(x) = \chi_M(x)$ (where $\chi_M(x)$ is the characteristic function of the set M). In this case we shall say of the predicate $P(x)$ that it solves the problem of membership in the set M with frequency $1-\varepsilon$ on infinitely many initial segments.

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Theorem 1. *Whatever recursively enumerable set M and $\varepsilon > 0$ we take, the problem of membership in the set M is algorithmically solvable with frequency $1 - \varepsilon$ on infinitely many initial segments.*

Proof. Let M be an arbitrary recursively enumerable set and $\varepsilon > 0$. Put $M^{(n)} = \{x \mid x \in M \ \& \ x \leqslant n\}$. Denote by i_0 the greatest natural number for which, for infinitely many natural n, the inequality

$$(i_0 - 1) \cdot {}^1/_2 \varepsilon n \leqslant |M^{(n)}| < i_0 \cdot {}^1/_2 \varepsilon n$$

still holds (where $|M^{(n)}|$ is the cardinality of the set $M^{(n)}$). Taking into account

* An analogous question also arises in the decoding of automata (without the use of a priori information), and in ([1]) it is, in a certain sense, answered positively.

** Here use will be made of ideas similar to those used in the proof of Theorem 4 from ([2]) and, apparently, also of Theorem 4.4 from ([3]). In particular, Theorem 4 of the work ([2]) follows easily from Theorem 1.

that the set $\{n \mid |M^{(n)}| \geq i_0\}$ is finite, we easily obtain that the set $\{n \mid (i_0 - 1) \cdot {}^1/_2 \varepsilon n \leq |M^{(n)}| < i_0 \cdot {}^1/_2 \varepsilon n\}$ is recursively enumerable. Moreover, this set is infinite. Therefore there exists a general recursive function $\eta(s)$ such that, for all natural s: a) $(i_0 - 1) \cdot {}^1/_2 \varepsilon \eta(s) \leq |M^{(\eta(s))}| < i_0 \cdot {}^1/_2 \varepsilon \eta(s)$; b) $\eta(s+1) \geq 2\eta(s)$.

Using the function $\eta(s)$, we shall now construct the desired predicate $P(x)$ (more precisely, a predicate $P(x)$ for which, for every s, the inequality $[M, P, \eta(s)]/\eta(s) \geq 1 - \varepsilon$ holds). First note that, using the function $\eta(s)$, for every s one can effectively construct a set R_s such that: a) $R_s \subset M^{(\eta(s))}$; b) $|M^{(\eta(s))} \setminus R_s| < {}^1/_2 \varepsilon \eta(s)$.

This can be done, for example, in the following way: take $\eta(s)$ copies of Turing machines computing the partial characteristic function of the set M, write on their tapes respectively the numbers $1, 2, \dots, \eta(s)$, start them running, and wait until $\lfloor (i_0 - 1)\varepsilon/2\, \eta(s) \rfloor$ machines halt* (the fact that this will happen eventually follows from property a) of the function $\eta(s)$); the numbers that were written on the tapes of these $\lfloor (i_0 - 1) \cdot {}^1/_2 \varepsilon \eta(s) \rfloor$ machines (for them the partial characteristic function is defined, hence they belong to M) will form the desired set R_s. Put $R_1^* = R_1$, $R_s^* = \{x \mid x \in R_s \ \& \ x \geq \eta(s-1)\}$, $s = 2, 3, \dots$, and

$$R = \bigcup_{s=1}^{\infty} R_s^*.$$

Let $\alpha(s)$ be the number of elements of the set $(M \setminus R) \cup (R \setminus M)$ not exceeding $\eta(s)$. Since $R \subseteq M$ and

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$$\{x \mid x \in R \ \& \ x \leq \eta(s)\} = \bigcup_{i=1}^{s} R_i^*,$$

we have

$$\alpha(s) = \left| M^{(\eta(s))} \setminus \bigcup_{i=1}^{s} R_i^* \right|.$$

Using $\eta(i+1) \geq 2\eta(i)$ and $|M^{(\eta(i))} \setminus R_i| < {}^1\!/_2 \varepsilon \eta(i)$, it is not hard to show (by induction on s) that

$$\left| M^{(\eta(s))} \setminus \bigcup_{i=1}^{s} R_i^* \right| < \varepsilon \eta(s).$$

Thus, $\alpha(s) < \varepsilon\eta(s)$. Now choose as the desired predicate $P(x)$ the characteristic function of the set R. We obtain that $[M, P, \eta(s)]/\eta(s) = 1 - \alpha(s)/\eta(s) > 1 - \varepsilon$. Since R is a recursive set and, consequently, $P(x)$ is a general recursive predicate, the validity of Theorem 1 follows.

Above, in defining the notion of solvability of the membership problem with frequency $1-\varepsilon$, we proceeded from the idea that the natural numbers are taken in their natural order. However, of interest is also the more general case in which a certain general recursive function $\pi(s)$ is given and we first take $\pi(1)$, then $\pi(2), \pi(s), \ldots$, and try to solve the membership problem for these numbers. We shall say that the membership problem in a set M, under the numbering $\pi(s)$, is algorithmically solvable with frequency $1-\varepsilon$ on infinitely many initial segments if there exists a general recursive predicate $P(x)$ such that, for infinitely many natural n, the inequality $[M, P, \pi, n]/n \geq 1 - \varepsilon$ holds, where $[M, P, \pi, n]$ is the number of those x, not exceeding n, for which $P(\pi(x)) = \chi_M(\pi(x))$.

Since the preimage of a recursively enumerable set under a general recursive mapping $\pi(s)$ is also a recursively enumerable set, it follows from Theorem 1 that

Corollary. *Whatever the general recursive function $\pi(s)$, the recursively enumerable set M, and $\varepsilon > 0$ may be, the membership problem in the set M, under the numbering $\pi(s)$, is algorithmically solvable with frequency $1-\varepsilon$ on infinitely many initial segments.*

The following question also arises: can Theorem 1 be strengthened so that, in its formulation, the phrase "solvable with frequency $1-\varepsilon$ on infinitely many initial segments" is replaced by the phrase "solvable with frequency $1-\varepsilon$ on all initial segments" ? A negative answer to this question follows from Theorem 4 of paper ([4]).

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*$\lfloor l \rfloor$ is the smallest natural number $\geq l$.

3°. In Theorem 1 nothing is said about whether, from the number of a recursively enumerable set, one can effectively find the corresponding predicate $P(x)$. Therefore the following question arises. Suppose that some ε_0, $0 < \varepsilon_0 < 1$, and some numbering τ of recursively enumerable sets are fixed (τn is the set with number n). Is the following mass problem solvable: for any natural n, find the Kleene number of a general recursive predicate that solves the membership problem for the set τn with frequency $1-\varepsilon$ on infinitely many initial segments? In what follows we shall call such a mass problem the metaproblem $\mathcal{M}_{\varepsilon_0,\tau}$.

Theorem 2. *For any ε_0, $0 < \varepsilon_0 < 1$, and any principal numbering τ of recursively enumerable sets* [5]*, the metaproblem $\mathcal{M}_{\varepsilon_0,\tau}$ is algorithmically undecidable.*

Proof. Suppose, to the contrary, that $\mathcal{M}_{\varepsilon_0,\tau}$ is algorithmically decidable. Consider a universal recursive predicate $F(s,x)$ corresponding to some principal numbering of one-place recursive predicates (the principal numbering is needed here in order that the fixed-point theorem apply). By an F-number of a recursively enumerable set M we shall mean any s for which $M = \{x \mid F(s,x) = 1\}$. It is obvious that there exists a general recursive function that reduces the F-numbering to the principal numbering τ. Therefore, from the assumption that $\mathcal{M}_{\varepsilon_0,\tau}$ is algorithmically decidable it follows that there must exist a two-place general recursive predicate $P(s,x)$ which assigns to any recursively enumerable set M_s with F-number s the one-place predicate $P_s(x) = P(s,x)$ such that for infinitely many n one has $[M_s, P_s, n]/n \geq 1-\varepsilon$. Consider, instead of the predicate $P(s,x)$, the opposite predicate $\overline{P}(s,x)$. By the fixed-point theorem we obtain that there exists a number a for which $F(a,x) = \overline{P}(a,x)$. Consider the set M_a with F-number a. We obtain

$$M_a = \{x \mid F(a,x) = 1\} = \{x \mid \overline{P}(a,x) = 1\} = \{x \mid P(a,x) = 0\}.$$

Thus, for every x, the value of the predicate $P_a(x) = P(a,x)$ is opposite to the value of the characteristic function $\chi_{M_a}(x)$ of the set M_a. Therefore, for every n, $[M_a, P_a, n] = 0$ and, consequently, $[M_a, P_a, n]/n < 1-\varepsilon$. The contradiction obtained proves the theorem.

In view of the negative result of Theorem 2, the following question is of interest: perhaps with frequency $1-\varepsilon$ we nevertheless can solve the problem $\mathcal{M}_{\varepsilon_0,\tau}$? We shall say that the metaproblem $\mathcal{M}_{\varepsilon_0,\tau}$ is algorithmically decidable with frequency $1-\varepsilon$ on infinitely many initial segments if there exists a general recursive function $F(x)$ such that for infinitely many natural n the inequality $[\mathcal{M}_{\varepsilon_0,\tau}, F, n]/n \geq 1-\varepsilon$ holds, where $[\mathcal{M}_{\varepsilon_0,\tau}, F, n]$ is the number of those natural x, not exceeding n, for which $F(x)$ is the Kleene number of a general recursive predicate solving the membership problem for the set τx with frequency $1-\varepsilon_0$ on infinitely many initial segments. The following theorem gives a positive answer to the question posed.

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Theorem 3. *Whatever ε_0, $0 < \varepsilon_0 < 1$, a computable numbering τ of recursively enumerable sets, and $\varepsilon > 0$ may be, the metaproblem $\mathcal{M}_{\varepsilon_0,\tau}$ is algorithmically decidable with frequency $1 - \varepsilon$ on infinitely many initial segments.*

The proof of Theorem 3 is based on the same idea as was used in the proof of Theorem 1. However, technically the proof of Theorem 3 is considerably more complicated.

4°. In connection with Theorem 1 there arises still the following question (in a certain sense the opposite of the question of the preceding item): is it true that for any fixed recursively enumerable set M there exists an algorithm which, for any rational $\varepsilon > 0$, constructs the Kleene number of a predicate solving the membership problem in the set M with frequency $1-\varepsilon$ on infinitely many initial segments?

It can be shown, by reducing to a contradiction with Theorem 3 of ([2]), that the answer to this question is negative. However, the question of whether the given mass problem is decidable with frequency $1 - \varepsilon$ remains open.

5°. Alongside the complexity of programs recognizing, among the natural numbers not exceeding n, membership in the set M ([2]), one may also consider the complexity of programs that do the same only with frequency $1 - \varepsilon$. This complexity (denote it by $K(M, n, \varepsilon)$) may be defined as follows: let $A(p, x)$ be a partial recursive function (a "programming method") for which the minimal complexity is obtained, up to an additive constant (see ([2])), and let $A_p(x) = A(p, x)$; then $K(M, n, \varepsilon)$ is the minimal length of numbers p such that $[M, A_p, n]/n \geq 1 - \varepsilon$, where $[M, A_p, n]$ is the same as in 2°. It is not hard to see that for any recursively enumerable set M and any natural n, $K(M, n, \varepsilon) \leq \log_2 1/\varepsilon + C$, where C is a constant independent of n and ε. The question of whether this upper bound (as a function of ε) is final remains open.

By analogy with $K^t_{\mathfrak{A}}(M; n)$ from ([2]), one may also introduce $K^t_{\mathfrak{A}}(M, n, \varepsilon)$, characterizing the complexity of programs recognizing membership, among the natural numbers not exceeding n, in the set M with frequency $1 - \varepsilon$ under the restriction $t(x)$ on the admissible difficulty of processing programs. In that case, using Theorem 3 of ([2]), one can show that there exists a recursively enumerable set M with the following property: for every general recursive function $t(x)$ there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \leq \varepsilon_0$ and every natural n, one has $K^t_{\mathfrak{A}}(M, n, \varepsilon) \geq C_{t,\varepsilon} n$, where $C_{t,\varepsilon}$ is a positive constant independent of n (but dependent on t and ε). This shows that, for sufficiently small ε, the estimate from Theorem 3 ([2]) cannot be improved in order of magnitude by replacing ordinary decidability with decidability with frequency $1 - \varepsilon$.

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Note: Figure translations are in progress. See original paper for figures.

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