ON THE OSCILLATION OF SOLUTIONS

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Abstract

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ON THE OSCILLATION OF SOLUTIONS OF VECTOR DIFFERENTIAL EQUATIONS

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1. Numerous works are devoted to the question of oscillation and nonoscillation of solutions of ordinary differential equations. For the vector case, various definitions of oscillation of systems are available in the literature. In ([1]) one such definition is given (for the case of a system with two degrees of freedom), along with conditions for oscillation of the system and the definition of an oscillating vector-function (all its components oscillate); however, no theorems are given on the existence of such solutions for an oscillating system. For certain special cases of first-order linear systems $u'(t) = A(t)u(t)$, such criteria are given in ([5,6]). However, the definition of an oscillating vector-function as a function all of whose components oscillate is, in our opinion, hardly the most reasonable multidimensional analogue of the well-known scalar concept, since this property is not invariant under rotation of the coordinate system.

In the present paper we give the notions of an oscillating and a strictly nonoscillating function with values in a finite-dimensional E_n or infinite-dimensional real Hilbert space H, and we present necessary, sufficient, as well as necessary and sufficient conditions for oscillation of solutions of one class of vector differential equations.

2. **Definition 1.** A vector-function $u(t)$, whose values lie in H for all t and which is continuous on this interval, is called **oscillating** if the scalar functions $(u(t), h)$ have an infinite number of zeros for any $h \in H$, and **nonoscillating** otherwise.

Definition 2. We shall call the vector-function $u(t)$ **strictly nonoscillating** if there exist $t_1 \geq 0$ and $h \in H$ such that

$$(u(t), h)/\|u(t)\| \geq a > 0 \quad \text{for } t \geq t_1 \tag{1}$$

(this means that the vectors $h \in H$ for which $(u(t), h) = 0$ for $t \geq t_1$ fill some solid cone).

In what follows we shall consider functionals $F(t,u)$, defined and continuous jointly in the variables on $\{t \geq 0,\ u \in H\}$, satisfying the following **conditions (F)**:

a) $F(t,u) \geq 0,\ F(t,0) \equiv 0$;

b) for any $t_0 > 0$ and $u \neq 0 \in H$, $\mathrm{mes}\{E_u \cap (t_0,\infty)\} > 0$, where $E_u = \{t : F(t,u) \neq 0\}$.

Definition 3. We shall say that the functional $F(t,u)$ is subject to the **K-condition on any ray** if, for every $v \neq 0 \in H$ and arbitrary $0 < \theta_0 < 1$, one can specify a constant $K = K(v,\theta_0) > 0$ such that for any $\theta_0 \leq \theta \leq 1$ and all $t \geq 0$,

$$F(t,\theta v) \leq K F(t,v). \tag{2}$$

Definition 4. We shall say that $F(t,u)$ is subject to the **M-condition** if, for every $v \neq 0 \in H$, there exists $M = M(v) > 0$ such that, for all $\|u\| \geq \|v\|$ and $t \geq 0$,

$$F(t,v) \leq M F(t,u). \tag{3}$$

If the constant M can be chosen the same for all ν, then we shall speak of a **uniform M-condition**.

It is not hard to see that if the M-condition is satisfied, then the K-condition is also satisfied.

3. Consider in H (or E_n) the differential equation

$$u''(t) + F(t,u(t))\,u(t)/\|u(t)\| = 0, \qquad t \geq 0. \tag{4}$$

It is natural to call it the equation of motion of a point under the action of a centrally acting force. This very class of equations turned out to be a reasonable generalization of the scalar equations $x'' + f(t,x) = 0$ in relation to our definition of oscillation.

In the following theorems we give necessary, sufficient, and also necessary and sufficient conditions either for the absence of nonoscillatory solutions, or for the absence of strictly nonoscillatory continuable solutions of equation (4). By continuable solutions we mean solutions defined on some (t_0,∞).

Theorem 1. *If $F(t,u)$ is subject to the K-condition on every ray and equation (4) has no strictly nonoscillatory solutions, then for every $h \neq 0 \in H$ one has*

$$\int^{\infty} tF(t,h)\,dt = \infty, \qquad \int^{\infty} F(t,th)\,dt = \infty. \tag{5}$$

Proof is carried out as follows. Suppose that for some $h \neq 0 \in H$

$$\int^{\infty} tF(t,h)\,dt < \infty.$$

Construct the integral equation

$$u(t) = \frac{h}{2} + t\int_t^{\infty} F(s,u(s))\frac{u(s)}{\|u(s)\|}\,ds + \int_{t_1}^{t} sF(s,u(s))\frac{u(s)}{\|u(s)\|}\,ds, \tag{6}$$

every solution of which satisfies (4), and it is shown, with the aid of Schauder' s fixed-point principle, that for sufficiently large t_1 equation (6) has a solution from $S \subset C(t_1,\infty;H)$, where

$$S = \{u(\cdot) \in C : \ u(t) = \varphi(t)h, \quad 1/2 \leq \varphi(t) \leq 1\},$$

i.e., a strictly nonoscillatory solution. The assumption that

$$\int^{\infty} F(t,th)\,dt < \infty$$

for some $h \neq 0 \in H$ is refuted analogously; only in (6) one must write th instead of h.

For the remaining theorems we give only their statements.

Theorem 2. *Let $F(t,u)\|u\|^{-1}$ be subject to the M-condition and suppose that for every $h \neq 0 \in H$ one has*

$$\int^{\infty} F(t,h)\,dt = \infty. \tag{7}$$

Then every continuable solution of (4) is oscillatory.

Theorem 3. *Let $F(t,u)$ be subject to the M-condition and suppose that for every $h \neq 0 \in H$ (7) holds.*

Then (4) has no strictly nonoscillatory solutions.

Theorem 4. *Suppose that the conditions of Theorem 3 are satisfied and that for every $h \neq 0 \in H$ there exist $q \neq 0 \in H$ and $R > 0$ such that, for sufficiently large t,*

$$F(t,h) \geq RF(t,tq). \tag{8}$$

Then condition (7) for the absence of strictly nonoscillatory solutions of equation (4) is not only sufficient but also necessary.

As was shown in Theorem 1, the divergence of the integral

$$\int_{t_0}^{t} sF(s,h)\,ds$$

for any $h \neq 0 \in H$ is a necessary condition for the oscillation of any continuable solution of (4). If one narrows the class of equations (4) and prescribes a sufficient rate of growth of this integral, then this condition can become sufficient, as the following two theorems show.

Theorem 5. *If $F(t,u)\|u\|^{-1}$ satisfies the M-condition and, for any $h \neq 0 \in H$, the condition*

$$\limsup_{t\to\infty}\left[\int_{t_0}^{t} sF(s,h)\,ds - A\ln t\right] = +\infty \tag{9}$$

is fulfilled for all A, then all solutions of (4) oscillate.

Theorem 6. *If $F(t,u)\|u\|^{-1}$ satisfies the uniform M-condition and, for any $h \neq 0 \in H$, condition (9) is fulfilled for $A = \frac{1}{4}M\|h\|$, then all solutions of (4) oscillate.*

Let, in particular, the functional $F(t,u)$ have the form $F(t,u) = a(t)\Phi(u)$. Then conditions (F) take the form

$$(\widetilde{F}): \quad a(t) \geq 0; \qquad \Phi(u) > 0 \text{ for } u \neq 0; \qquad \Phi(0) = 0.$$

In this case, as is not difficult to see, the M-condition becomes the requirement

$$\lim_{\|u\|\to\infty} \Phi(u) > 0,$$

while the uniform M-condition becomes the requirement of almost increase of the functional $\Phi(u)$: there exists $M > 0$ such that, for any $u_1, u_2 \in H$, $0 < \|u_1\| \leq \|u_2\|$, one has $\Phi(u_1) \leq M\Phi(u_2)$.

As for the K-condition, it is automatically fulfilled if $(\widetilde{F})$ is satisfied. The latter circumstance makes it possible to assert that Theorem 1 for the equation

$$u'' + a(t)\Phi(u)u/\|u\| = 0 \tag{10}$$

is already unimprovable (in the sense of extending the admissible class of continuous functionals Φ), since, having required $(\widetilde{F})$, one can no longer discard the requirement $\Phi(u) > 0$ for $u \neq 0$ without violating the condition of absence of strictly oscillating solutions, as is not difficult to show.

4. The results presented, when applied to the scalar case, apparently are also new (cf. ([2–4])). We give two possible applications of the theorems to the scalar case. For compactness of formulation we take the equation in the particular form (10):

$$x'' + a(t)f(x) = 0, \tag{11}$$

where $a(t) \geq 0$ and $f(x)$ are continuous functions.

Theorem 7. *If all solutions of (11) oscillate, then $f(x)x > 0$ for $x \neq 0$,*

$$\int^{\infty} ta(t)\,dt = \infty$$

and, for any $c \neq 0$,

$$\int^{\infty} a(t)|f(ct)|\,dt = \infty.$$

Theorem 8. *Let $f(x)x > 0,\ x \neq 0$, and let at least one of the quantities*

$$\sup_{x>0} f(x), \qquad \inf_{x<0} f(x)$$

be finite.

Then, for the oscillation of all solutions of (11), it is necessary, and when

$$\liminf_{x\to\infty} f(x) > 0 \quad \text{and} \quad \limsup_{x\to-\infty} f(x) < 0$$

also sufficient, that

$$\int^{\infty} a(t)\,dt = \infty.$$

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Note: Figure translations are in progress. See original paper for figures.