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Abstract

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FAMILIES OF HOMEOMORPHISMS EQUICONTINUOUS WITH RESPECT TO RELATIVE METRICS

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Among several directions in the study of the metric properties of topological mappings of various classes (conformal, quasiconformal, with bounded Dirichlet integral, and more general ones, plane and spatial), there is one that originates from the work of M. A. Lavrent' ev (^{1}), whose main task is to establish a two-sided (or at least one-sided) estimate of distortion by means of a specially introduced "relative" distance in domains, with respect to which the domains are completed. In concrete cases such estimates contain much information about the fundamental metric properties of the classes of mappings under study. An essential point in such investigations is the passage to estimates of the distortion of ordinary distances (more precisely, distances in the second metric, often in the metric of the ambient spaces) inside domains, although, of course, such estimates are often established directly, without explicit use of estimates of distortion of relative distances. Estimates inside domains give the properties of equicontinuity and openness of classes of mappings inside domains, which in a very general case is sufficient for establishing an analogue of the classical Carathéodory theorem on the connection between the concept of uniform convergence of a sequence of conformal mappings and the concept of convergence of a sequence of domains to a nondegenerate kernel; and such an analogue is the main instrument, for example, in proving existence theorems in the theory of quasiconformal mappings, stability theorems, etc.

The availability of estimates of distortion of relative distances, in addition to the natural solution of the question of boundary correspondence under mappings of domains, makes it possible to solve the principal problems concerning mappings of "closed" domains with variable boundaries.

Such investigations have been carried out for various classes of mappings; moreover, after the basic estimates of distortion of "relative" distances have been obtained, the arguments for deriving consequences from the estimates turn out in many respects to be similar. The question arises of identifying the information that is in general contained in such estimates, given a priori.

In the present paper we give a simple Theorem 1 stating that the existence of an equicontinuous two-sided estimate of distortion of distances in metric spaces is equivalent to the property of equicontinuity of the direct and inverse mappings. Next an answer is given (in the form of sufficient conditions) to the questions: 1) when estimates of distortion of "relative" distances imply estimates of distortion of the second metrics "inside" the spaces (Theorem 2); 2) under what conditions an analogue of Carathéodory' s theorem holds (Theorem 3).

It is possible to define rather clearly the necessary requirements on the metrics and on the mappings. These requirements turn out to be of one nature, and it is precisely they, explicitly or implicitly, that are present in all concrete investigations known to us.

I. On the distortion of distances.

1°. Consider two families $(D_i)_{i\in I}$, $D_i=(x^i)$, $(\Delta_i)_{i\in I}$, $\Delta_i=(y^i)$, of sets D_i and Δ_i of elements (points) of arbitrary nature. Suppose that on D_i, Δ_i, for each $i\in I$, two metrics are defined; on D_i: $r^\alpha=r^\alpha(x_1^i,x_2^i;D_i)$, $r^\beta=r^\beta(x_1^i,x_2^i;D_i)$, on Δ_i: $\rho^\alpha=\rho^\alpha(y_1^i,y_2^i;\Delta_i)$, $\rho^\beta=\rho^\beta(y_1^i,y_2^i;\Delta_i)$, so that we have 4 families of metric spaces: $(D_i^\alpha)_{i\in I}$, $(D_i^\beta)_{i\in I}$, $(\Delta_i^\alpha)_{i\in I}$, $(\Delta_i^\beta)_{i\in I}$, $D_i^\alpha=(D_i,r^\alpha)$, $D_i^\beta=(D_i,r^\beta)$, $\Delta_i^\alpha=(\Delta_i,\rho^\alpha)$, $\Delta_i^\beta=(\Delta_i,\rho^\beta)$. Let $(T_i)_{i\in I}$, $T_i:D_i\leftrightarrow\Delta_i$, be a family of one-to-one mappings.

For brevity, in what follows we shall omit the index in the notation of points of a space and the carrier in the notation of distances. Most often we shall also omit the symbol $i\in I$ in the notation of the families (D_i^α), (T_i), etc. The set I is assumed fixed in all arguments. θ denotes the empty set.

2°. Let $(\widehat{D}_i^\alpha)$ be a family of subsets of the spaces (D_i^α), $\widehat{D}_i^\alpha\subset D_i^\alpha$. The case $\widehat{D}_i^\alpha=\widehat{D}_i^\alpha$ is not excluded.

Definition 1. $(T_i)\in(a)_{\alpha\beta}$ on $(\widehat{D}_i^\alpha)$ (not all $\widehat{D}_i^\alpha=\theta$), if for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that for points $x_1\in\widehat{D}_i^\alpha$ and $x_2\in D_i^\alpha$, when $r^\alpha(x_1x_2)\le\delta(\varepsilon)$, one has $\rho^\beta[T_i(x_1),T_i(x_2)]\le\varepsilon$, and δ does not depend on $i\in I$.

Definition 2. $(T_i)\in(b)_{\alpha\beta}$ on $(\widehat{D}_i^\alpha)$, if for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that for each point $x\in\widehat{D}_i^\alpha$ the image of its spherical ε-neighborhood contains the spherical δ-neighborhood of the point $T_i(x)$ in Δ_i^β, and δ does not depend on $i\in I$ or on the choice of the point $x\in\widehat{D}_i^\alpha$.

Definition 3. $(T_i)\in(ab)_{\alpha\beta}$ on $(\widehat{D}_i^\alpha)$, if $(T_i)\in(a)_{\alpha\beta}$ and $(T_i)\in(b)_{\alpha\beta}$ on $(\widehat{D}_i^\alpha)$.

3°. Denote by $\overline{D}_i^\alpha$ the completion of D_i^α with respect to r^α.

Theorem 1. In order that there exist a number $a>0$ and two continuous, strictly monotonically increasing on $[0,a]$ functions $\varphi(\xi)$, $\psi(\xi)$, $\varphi(0)=\psi(0)=0$, such that for all points $x_1,x_2\in D_i^\alpha$, under the condition $r^\alpha(x_1x_2)\le a$, one has

$$\psi[r^\alpha(x_1x_2)] \le \rho^\alpha[T_i(x_1)T_i(x_2)] \le \varphi[r^\alpha(x_1x_2)] \tag{1}$$

(a, φ, ψ do not depend on $i \in I$), it is necessary and sufficient that two conditions hold: $(T_i) \in (a)_{\alpha\alpha}$ on (D_i^α), $(T_i^{-1}) \in (a)_{\alpha\alpha}$ on (Δ_i^α).

Under the conditions of this theorem each T_i is a homeomorphism $D_i^\alpha \leftrightarrow \Delta_i^\alpha$, extendable to a homeomorphism $\widetilde{T}_i : \overline{D}_i^\alpha \leftrightarrow \overline{\Delta}_i^\alpha$, and (1), obviously, is valid also on the completed spaces, so that $(\widetilde{T}_i) \in (a)_{\alpha\alpha}$ on $(\overline{D}_i^\alpha)$ and $(\widetilde{T}_i^{-1}) \in (a)_{\alpha\alpha}$ on $(\overline{\Delta}_i^\alpha)$.

4°. Let $\eta > 0$. Put $\partial\overline{D}_i^\alpha = \overline{D}_i^\alpha \setminus D_i^\alpha$ and

$$D_i^{(\alpha)}(\eta) \equiv D_i^{(\alpha)}(\eta, O_i) = \begin{cases} D_i, & \text{if } \partial\overline{D}_i^\alpha = \theta, \\ \{x \in D_i : \ r^\alpha(x, \partial\overline{D}_i^\alpha) \ge \eta\}_{O_i}, & \text{if } \partial\overline{D}_i^\alpha \ne \theta, \end{cases}$$

where $\{...\}_{O_i}$ is the connected component of the set $\{...\}$ containing the point $O_i \in D_i$. The sets $D_i^{(\beta)}(\eta) \equiv D_i^{(\beta)}(\eta, O_i)$ are defined analogously, with the points O_i the same as above.

Definition 4. $D_i^\alpha \in (C)_\alpha$, if, considering $D_i^{(\alpha)}(\eta)$ as a subset of the space D_i^α, we have

$$D_i^\alpha \subset \bigcup_{\eta > 0} D_i^{(\alpha)}(\eta).$$

Let $x = e_i(x) : D_i^\alpha \leftrightarrow D_i^\beta$, $y = E_i(y) : \Delta_i^\alpha \leftrightarrow \Delta_i^\beta$ be the natural mappings, and e_i^{-1} and E_i^{-1} the inverse mappings.

Theorem 2. Let $(T_i) \in (a)_{\alpha\alpha}$ on (D_i^α), $(T_i^{-1}) \in (a)_{\alpha\alpha}$ on (Δ_i^α), $T_i(O_i) = O_i'$. Suppose that for all $i \in I$: $\overline{D}_i^\alpha, \overline{D}_i^\beta, \overline{\Delta}_i^\alpha, \overline{\Delta}_i^\beta$ are compact, $D_i^\alpha \in (C)_\alpha$, $(e_i) \in (ab)_{\alpha\beta}$ on $[D_i^{(\alpha)}(\eta)]$, $(e_i^{-1}) \in (ab)_{\beta\alpha}$ on $[D_i^{(\beta)}(\eta)]$,

$(E_i) \in (ab)_{\alpha\beta}$ on $[\Delta_i^{(\alpha)}(\eta)]$, $(E_i^{-1}) \in (ab)_{\beta\alpha}$ on $[\Delta_i^{(\beta)}(\eta)] = \Delta_i^{(\beta)}(\eta, O_i')$ for all $\eta > 0$. Then: 1) there exists a number $a(\eta) > 0$ and continuous, strictly monotonically increasing on $[0, a(\eta)]$ functions $\Phi_j^\eta(\xi)$, $\overline{\Phi}_j^\eta(\xi)$, $\Phi_j^\eta(0) = \overline{\Phi}_j^\eta(0) = 0$ $(j = 1, 2)$, such that

$$\Phi_1^\eta[r^\beta(x_1x_2)] \le \rho^\beta[T_i(x_1), T_i(x_2)] \le \Phi_2^\eta[r^\beta(x_1x_2)]$$

for all $x_1 \in D_i^{(\beta)}(\eta)$, $x_2 \in D_i^\beta$, $i \in I$, $D_i^{(\beta)}(\eta) \ne 0$, under the condition

$$r^\beta(x_1x_2) \le a(\eta),$$

and

$$\overline{\Phi}_1^{\eta}[\rho^{\beta}(y_1y_2)] \le r^{\beta}(T_i^{-1}(y_1)T_i^{-1}(y_2)) \le \overline{\Phi}_2^{\eta}[\rho^{\beta}(y_1y_2)]$$

for all $y_1 \in \Delta_i^{(\beta)}(\eta)$, $y_2 \in \Delta_i^{\beta}$, $i \in I$, $\Delta_i^{(\beta)}(\eta) \neq 0$, under the condition

$$\rho^{\beta}(y_1y_2) \le a(\eta).$$

The number $a(\eta)$ does not depend on $i \in I$. 2) $(T_i) \in (ab)_{\beta\beta}$ on $[D_i^{(\beta)}(\eta)]$ and $(T_i^{-1}) \in (ab)_{\beta\beta}$ on $[\Delta_i^{(\beta)}(\eta)]$.

II. An analogue of Carathéodory’ s theorem.

5°. Let $M^{\beta} = (M, r^{\beta})$, $M = (x)$, $r^{\beta} = r^{\beta}(x_1x_2)$, $N^{\beta} = (N, \rho^{\beta})$, $N = (y)$, $\rho^{\beta} = \rho^{\beta}(y_1y_2)$ be two metric spaces, connected, locally connected, locally compact, and separable. Let (D_n^{β}), $O \in D_n^{\beta} \subset M^{\beta}$, $D_n^{\beta} = (D_n, r^{\beta})$, $D_n \subset M$; (Δ_n^{β}), $O' \in \Delta_n^{\beta} \subset N^{\beta}$, $\Delta_n^{\beta} = (\Delta_n, \rho^{\beta})$, $\Delta_n \subset N$ $(n = 1, 2, ...)$, be two sequences of domains, and suppose all domains of the first sequence contain the fixed ball $S^{\beta}(r_0) : r^{\beta}(O, x) < r_0$.*

Consider also two further sequences of metric spaces: (D_n^{α}), $D_n^{\alpha} = (D_n, r_n^{\alpha})$, $r_n^{\alpha} = r_n^{\alpha}(x_1x_2)$, and (Δ_n^{α}), $\Delta_n^{\alpha} = (\Delta_n, \rho_n^{\alpha})$, $\rho_n^{\alpha} = \rho_n^{\alpha}(y_1y_2)$, whose carriers D_n and Δ_n are the same as above.**

6°. Consider the sequence (D_n^{β}) of domains of M^{β} under the assumptions of item 5°. Let D_0^{β} be a domain in M^{β} that is the union of all domains $D^{\beta} \subset M^{\beta}$, $D^{\beta} \ni O$, having the following property: every point $x \in D^{\beta}$, together with some neighborhood of it in M^{β}, belongs to D_n^{β} for $n \ge \bar{n}$. D_0^{β} is called the nondegenerate kernel of the sequence (D_n^{β}) with respect to the point O. The sequence (D_n^{β}) converges to D_0^{β} if every infinite subsequence of it has kernel D_0^{β} (with respect to the point O).

7°. **Theorem 3** (an analogue of Carathéodory’ s theorem). Let M^{β}, N^{β} be connected, locally connected, locally compact, and separable metric spaces, (D_n^{β}), (Δ_n^{β}) sequences of domains of the spaces M^{β} and N^{β}, respectively; D_n^{β} contains the ball $S^{\beta}(r_0)$, $\Delta_n^{\beta} \ni O'$, (T_n) a sequence of one-to-one mappings $D_n \leftrightarrow \Delta_n$, $T_n(O) = O'$ for all n. In addition:

1) $\overline{D}_n^{\alpha}, \overline{D}_n^{\beta}, \overline{\Delta}_n^{\alpha}, \overline{\Delta}_n^{\beta}$ are compact for all n;
2) $(e_n) \in (ab)_{\alpha\beta}$ on $[D_n^{\alpha}(\eta)]$, $(e_n^{-1}) \in (ab)_{\beta\alpha}$ on $[D_n^{(\beta)}(\eta)]$, $(E_n) \in (ab)_{\alpha\beta}$ on $[\Delta_n^{(\alpha)}(\eta)]$, $(E_n^{-1}) \in (ab)_{\beta\alpha}$ on $[\Delta_n^{(\beta)}(\eta)]$ for each $\eta > 0$, if the corresponding sequences of sets are nonempty***;
3) $(T_n) \in (a)_{\alpha\alpha}$ on (D_n^{α}), $(T_n^{-1}) \in (a)_{\alpha\alpha}$ on (Δ_n^{α}).

Then from (n) one can select an infinite subsequence (n') such that:

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* Thus the metrics r^β and ρ^β are induced in D_n^β and Δ_n^β from M^β and N^β and, unlike in the preceding considerations, do not depend on n.

** The metrics r_n^α and ρ_n^α in general depend on n. It is more convenient for us to emphasize this here, and therefore we supply the notation with indices.

*** $D_n^{(\alpha)}(\eta)$, $D_n^{(\beta)}(\eta)$ are formed with respect to the point O, $\Delta_n^{(\alpha)}(\eta)$, $\Delta_n^{(\beta)}(\eta)$ with respect to the point O'.

a) $(D_{n'}^\beta)$ converges as to a kernel to the domain $D_0^\beta \supset S^\beta(r_0)$, $(\Delta_{n'}^\beta)$ converges as to a kernel to the domain $\Delta_0^\beta \ni O'$;

b) $(\widetilde{T}_{n'})$ converges uniformly inside D_0^β to the homeomorphism $T_0 : D_0^\beta \leftrightarrow \Delta_0^\beta$, $(\widetilde{T}_{n'}^{-1})$ converges uniformly inside Δ_0^β to T_0^{-1};

c) for any points $x_1x_2 \in D_0^\beta$ and $y_1y_2 \in \Delta_0^\beta$ the limits exist

$$\lim_{n'\to\infty} r_{n'}^\alpha(x_1x_2) = r_0^\alpha(x_1x_2), \qquad \lim_{n'\to\infty} \rho_{n'}^\alpha(y_1y_2) = \rho_0^\alpha(y_1y_2),$$

and r_0^α and ρ_0^α will be metrics, and the kernels may be regarded as the metric spaces D_0^α and Δ_0^α. For $\widetilde{T}_0$ and $\widetilde{T}_0^{-1}$ the inequalities (1) of Theorem 1 are satisfied, so that $\widetilde{T}_0 \in (a)_{\alpha\alpha}$ on $\widetilde{D}_0^\alpha$ and $\widetilde{T}_0^{-1} \in (a)_{\alpha\alpha}$ on $\widetilde{\Delta}_0^\alpha$.

The proof is carried out with the aid of Theorem 2 and arguments analogous to the arguments in (^{2}), § 18, in the proof of Theorem 21.

Remark 1. Theorem 21, § 18, in (^{2}) also gives an analogue of Carathéodory’ s theorem for homeomorphisms of domains belonging to arbitrary compacta. However, the requirements imposed there on the mappings are such that they sharply narrow the class of sequences of domains admissible for consideration. For example, the theorem does not completely cover the case of Carathéodory’ s theorem on conformal mappings. This is explained by the fact that in definitions $13'$, $14'$, § 18 (^{2}), which describe the assumed properties of mappings, arbitrary compacta internal to the domains are considered, whereas, in order to remove the indicated narrowing of the class of sequences of domains, one must consider not arbitrary compacta, but continua containing the origin, which is what is done everywhere subsequently in (^{2}) in the study of plane mappings. However, if definitions $13'$ and $14'$ are changed in this way, then the proof of Lemma 13, § 18 in (^{2}) does not go through. Additional restrictions on the containing spaces are needed. Our Theorem 3 is free from this shortcoming and goes somewhat beyond the theorem from (^{2}) in the sense of the requirements on the mappings (the properties of mappings postulated in (^{2}) are consequences of the assumptions of Theorem 3).

Remark 2. Theorem 3 can be formulated not for sequences, but for an arbitrary family of mappings (in the spirit of (^{3})). Probably, results analogous to ours can also be obtained for mappings of uniform structures.

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CITED LITERATURE

[1] M. A. Lavrent' ev, DAN, 4, 207 (1936).
[2] G. D. Suvorov, *Families of Plane Topological Mappings*, Novosibirsk, 1965.
[3] V. P. Lufarenko, Dopov. AN Ukrainian SSR, No. 1 (1970).

Note: Figure translations are in progress. See original paper for figures.