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Abstract

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ON THE MOTION OF THE SPIN OF PARTICLES IN A STORAGE RING WITH AN ARBITRARY FIELD

In studying the behavior of particle polarization in accelerators, one usually restricts oneself to the case of a magnetic field that is almost constant in direction ([6–9]). In the present work we set forth general results of an investigation of spin motion in storage rings (accelerators) with an arbitrary electromagnetic field, which are of practical interest.

1. Before turning directly to the subject of this paper, we give a simple and physically transparent derivation of the spin equations for particles with arbitrary spin. In the quasiclassical approximation the equation for the mean spin vector $\langle \hat{\mathbf{s}} \rangle = \xi$ in the particle rest frame is obtained from the known covariant equations of the magnetic moment ([1,2]). In this procedure, however, the physical transparency of the equations for ξ proves to be obscured. In fact, there exists a very simple derivation of the equations without the intermediate use of the polarization 4-vector.

In an inertial coordinate system coinciding with the proper one at the instant of time t (the i-system), over an interval of proper time $d\tau = dt/\gamma$ the vector ξ acquires an increment ([2,3])

$$d\xi = [\mu \mathbf{H}]\, d\tau,$$

where μ is the magnetic moment of the particle; $\mathbf{H}$ is the magnetic field in the rest frame. However, after a time dt the proper frame (obtained from the laboratory frame by a Lorentz transformation) turns out to be rotated relative to the i-system. The angle of rotation $d\varphi$ can be found from simple considerations.

Let $d\alpha = \frac{1}{v^2}[\mathbf{v}\, d\mathbf{v}]$ be the angle of rotation of the velocity in the laboratory frame. Then the proper frame, in accordance with its definition, will rotate relative to the new direction of the velocity $\mathbf{v} + d\mathbf{v}$ through the angle $-d\alpha$. At the same time the direction $(\mathbf{v} + d\mathbf{v})$ in the i-system makes, with the direction $\mathbf{v}$, the angle $\gamma\, d\alpha$. Indeed, these directions in the i-system are obtained by projecting the corresponding directions of the laboratory frame onto the plane

$\tau = \text{const}$. This procedure corresponds precisely to the definition of the angle in the i-system between two "rods" at rest in the laboratory frame. Consequently, the angle of rotation is

$$d\varphi = \gamma\, d\alpha - d\alpha = \frac{\gamma - 1}{v^2}[\mathbf{v}\, d\mathbf{v}].$$

Thus, the desired equation is:

$$\dot{\xi} = \frac{1}{\gamma}\frac{d\xi}{d\tau} - [\dot{\varphi}\xi] = \frac{1}{\gamma}[\mu\mathbf{H}\,] + \frac{\gamma - 1}{v^2}[\xi[\mathbf{v}\dot{\mathbf{v}}]]. \tag{1}$$

The second term in (1), not connected with the magnetic moment of the particle, is a consequence of the relativistic kinematics of rotation.* The idea of this derivation is contained in the old works of Thomas ([5]) ("Thomas half").

* For $\mu = 0$, in the rest frame rotating together with the velocity, after one revolution around a circle the spin turns about the direction $[\mathbf{v}\dot{\mathbf{v}}]$ not through the angle -2π, but through the angle $-2\pi\gamma$, as a consequence of the Lorentz "stretching of the circumference" ([4]) with the radius preserved.

Using the equations of motion of the particle and the expression for $\mathbf{H}_c$ in terms of the fields in the l-system, we transform (1) to the form

$$\dot{\xi} = [\mathbf{W}_l\xi]; \qquad \mathbf{W}_l = \left(1 + \gamma\frac{q'}{q_0}\right)\frac{[\mathbf{v}\dot{\mathbf{v}}]}{v^2} - \frac{q}{\gamma}\frac{(\mathbf{H}\mathbf{v})}{v^2}\mathbf{v} + \frac{q}{\gamma^2 v^2}[\mathbf{E}\mathbf{v}], \tag{2}$$

where $q = \mu/s = q_0 + q' = e/m + q'$, q' is the anomalous part of the gyromagnetic ratio ($c = 1$). Equations (1), (2) coincide with the corresponding equations of works ([1,2]).

2. Let us formulate the problem of the motion of the spin of a particle in a storage ring with an arbitrary electromagnetic field, assuming the existence of a closed (periodic, equilibrium) orbit. We shall consider the energy of the particle on the equilibrium trajectory to be constant. We write the angular velocity $\mathbf{W}_l$ in the form

$$\mathbf{W}_l = \mathbf{W}_s + \mathbf{w}; \qquad \mathbf{W}_s(\theta) = \mathbf{W}_s(\theta + 2\pi); \qquad \theta = \omega_s t, \tag{3}$$

where ω_s is the equilibrium circulation frequency of the particle; $\mathbf{W}_s$ is the value of $\mathbf{W}_l$ on the equilibrium trajectory; $\mathbf{w}$ is an addition associated with the deviation of the particle from the periodic orbit.

3. Let us turn to the study of the spin motion of equilibrium particles, satisfying the equation

$$\dot{\xi} = [\mathbf{W}_s \xi]. \tag{4}$$

The general solution of (4) can be represented as a combination of three linearly independent solutions $\mathbf{x}_\alpha(\theta)$ ($\alpha = 1, 2, 3$). Since the scalar product of any two solutions of (4) is conserved,

$$\frac{d}{dt}(\xi_a \xi_b) = 0, \tag{5}$$

the basic solutions can always be chosen to be constantly orthogonal:

$$\mathbf{x}_\alpha \mathbf{x}_\beta \equiv \delta_{\alpha\beta}; \qquad \alpha, \beta = 1, 2, 3. \tag{6}$$

Having specified some basis (6), we shall seek solutions (4) having the property

$$\xi(\theta + 2\pi) = \lambda \xi(\theta), \qquad \lambda = \text{const}.$$

Expanding ξ in the basis $\mathbf{x}_\alpha$, we obtain a system of equations for determining the projections $\xi_\alpha = \xi \mathbf{x}_\alpha$:

$$\sum_{\beta=1}^{3} (\lambda \delta_{\alpha\beta} - \Lambda_{\alpha\beta}) \xi_\beta = 0; \qquad \Lambda_{\alpha\beta} = \mathbf{x}_\alpha(\theta) \mathbf{x}_\beta(\theta + 2\pi).$$

In view of (5), the matrix Λ does not depend on time, since $\mathbf{x}_\beta(\theta + 2\pi)$ is a solution of (4) by virtue of the periodicity of $\mathbf{W}_s(\theta)$. A nonzero solution ξ_α exists if

$$|\lambda \delta_{\alpha\beta} - \Lambda_{\alpha\beta}| = \mathrm{Det}(\lambda I - \Lambda) = 0.$$

Since Λ, by virtue of (5), is a rotation matrix, $\lambda = 1$ is its eigenvalue, corresponding to a periodic solution

$$\mathbf{n}(\theta) = \mathbf{n}(\theta + 2\pi); \qquad \mathbf{n}^2 = 1.$$

The other two eigenvalues are found from the relations

$$\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{Sp}\, \Lambda; \qquad \lambda_1 \lambda_2 \lambda_3 = |\Lambda| = 1.$$

Hence

$$\lambda_2 = \lambda_3^* = e^{-2\pi i \nu}; \qquad \cos 2\pi\nu = \tfrac{1}{2}(\mathrm{Sp}\, \Lambda - 1).$$

It follows from (5) that $\cos 2\pi\nu$ does not depend on the choice of the basis $\mathbf{x}_\alpha$. The eigen-solutions $\mathbf{n}$; η, η^*, corresponding to the eigenvalues 1; $e^{-2\pi i\nu}$, $e^{2\pi i\nu}$, are orthogonal if $\cos 2\pi\nu \neq 1$; in this case the periodic solution is unique. In the case of exact resonance $\cos 2\pi\nu = 1$, degeneracy occurs, and any solution for the spin is periodic.

In what follows we shall consider the spin motion with respect to the following periodic basis:

$$\{e_\alpha\} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}); \qquad \mathbf{e}_1 + i\mathbf{e}_2 = \eta e^{i\nu\theta} \equiv \mathbf{e}. \tag{7}$$

The general solution of (4) can be written in the form

$$\xi(\theta) = \xi_n \mathbf{n} + \xi_\perp \operatorname{Re}(\mathbf{e} \cdot e^{-i\Psi}); \qquad \xi_n = \text{const}; \qquad \xi_\perp = \sqrt{s^2 - \xi_n^2};$$
$$\Psi = \nu\omega_s. \tag{8}$$

Thus, the motion of the spin on a periodic orbit proceeds as follows. There exists a certain periodic direction $\mathbf{n}(\theta)$, having the meaning of the direction of polarization of the periodic solution, around which the spin rotates while preserving its projection onto this direction. Further, independently of the point of observation and of the initial conditions, over one period of motion along the orbit the spin turns about $\mathbf{n}$ through the same angle $2\pi\nu$.

It is practically important that the structure of $\mathbf{W}_s$ makes it possible, at a prescribed point of the orbit, to create the required orientation of $\mathbf{n}$ relative to the velocity and the field.

4. Let us consider the question of the stability of the periodic solution with respect to variations of $\mathbf{W}_s$, which may be connected both with deviations of the real field and closed orbit from the ideal (design) ones and with changes of the parameters (for example, the energy) determining $\mathbf{W}_s$. In the linear approximation the variation $\delta\mathbf{n}$ satisfies the equation

$$\frac{d}{dt}\delta\mathbf{n} = [\mathbf{W}_s \delta\mathbf{n}] + [\delta\mathbf{W}_s \mathbf{n}].$$

A periodic solution of this equation is

$$\delta\mathbf{n} = \operatorname{Re}\mathbf{e} \sum_k \frac{(\delta\mathbf{W}_s \mathbf{e}^*)_k}{(\nu - k)\omega_s} e^{-ik\theta},$$

where

$$(\delta\mathbf{W}_s \mathbf{e}^*) = \sum_k (\delta\mathbf{W}_s \mathbf{e}^*)_k e^{-ik\theta}.$$

As is seen, the periodic solution $\mathbf{n}$ becomes very sensitive to small changes of $\mathbf{W}_s$ near the resonances $\nu = k$. This is the physical meaning of the above-noted indeterminacy of $\mathbf{n}$ at exact resonance.

5. The motion of the spin of particles moving near the equilibrium trajectory is conveniently considered with respect to the periodic basis (7). The angular velocity of spin rotation in this system is

$$\mathbf{W} = \mathbf{W}_l - \mathbf{W}_b = \mathbf{W}_l - \frac{1}{2}\sum_{\alpha=1}^{3}[\mathbf{e}_\alpha \dot{\mathbf{e}}_\alpha] = \nu\omega_s\mathbf{n} + \mathbf{w},$$

where $\mathbf{W}_b$ is the angular velocity of rotation of the basis trihedron.

In the system (7), the spin equations (2) have the form

$$\dot{\xi} = [\mathbf{W}\xi]. \tag{9}$$

Equation (9) can be written in Hamiltonian variables–the projection ξ_n and the phase of rotation Ψ about $\mathbf{n}$ (see (8)):

$$\dot{\xi}_n = -\frac{\partial\mathcal{H}}{\partial\Psi} = w_\perp\xi_\perp\sin(\Psi-\delta);$$

$$\dot{\Psi} = \frac{\partial\mathcal{H}}{\partial\xi_n} = \nu\omega_s + \mathbf{wn} - w_\perp\frac{\xi_n}{\xi_\perp}\cos(\Psi-\delta), \tag{10}$$

where

$$\mathcal{H} = \mathbf{W}\xi = (\nu\omega_s + \mathbf{wn})\xi_n + w_\perp\xi_\perp\cos(\Psi-\delta), \qquad w_\perp e^{i\delta} = \mathbf{we}.$$

For $\mathbf{w} = 0$, the solution (9), (10) coincides with (8).

Since in deriving (9), (10) no specific structure of the electromagnetic field was used, equations of the same form are obtained, in particular, for motion in an almost homogeneous magnetic field. Therefore the main conclusion of works ([6]–[9]) on the study of the stability of polarization in an almost homogeneous field can be extended to the general case as well: the motion of the spin near the periodic solution can become unstable only near resonances

$$\nu = k + \sum_i k_i\nu_i,$$

where ν_i are the frequencies of the betatron and energy oscillations of the particle.

6. According to papers [10–13], radiative polarization occurs in a magnetic field. In the ultrarelativistic case the degree of equilibrium polarization in a homogeneous field is equal to $8/5\sqrt{3}$. In paper [13] an equation was obtained for the ensemble-averaged polarization, taking into account radiation processes in an arbitrary external field ($s = 1/2$):

$$\dot{\xi} = [\mathbf{W}_\pi \xi] - \frac{1}{T}\left[\xi - \frac{2}{9}(\xi\mathbf{v})\mathbf{v} + \frac{4}{\xi\sqrt{3}}\frac{[\mathbf{v}\dot{\mathbf{v}}]}{|\dot{\mathbf{v}}|}\right];$$

$$\frac{1}{T} = \frac{5\sqrt{3}}{8}\alpha\frac{\hbar^2}{m^2}\gamma^5|\dot{\mathbf{v}}|^3; \qquad \alpha = \frac{e^2}{4\pi\hbar} = \frac{1}{137}.$$

Since the radiative term is small, the equation can be solved by the averaging method. Setting $\mathbf{w} = 0$, for $\mathbf{v} \neq \mathbf{k}$ we obtain the following result: the equilibrium polarization will be directed along $\mathbf{n}(\theta)$, and its degree is equal to

$$2\xi_n \underset{t\to\infty}{=} \frac{8}{5\sqrt{3}} \frac{\langle|\dot{\mathbf{v}}|^3 n_z\rangle}{\langle|\dot{\mathbf{v}}|^3(1 - {}^2/9 n_v^2)\rangle} \leq \frac{8}{5\sqrt{3}},$$

where $n_z = \mathbf{n}[\mathbf{v}\dot{\mathbf{v}}]/|\dot{\mathbf{v}}|$, $n_v = \mathbf{n}\mathbf{v}$, and the brackets $\langle ... \rangle$ denote averaging over the period of revolution of the particle.

7. The existence of stable periodic motion of the spin means that in a storage ring with an arbitrary electromagnetic field (provided a closed orbit exists) the beam polarization is stable to the same degree as in a storage ring with an almost constant-in-direction magnetic field. This opens broad possibilities for controlling polarization in storage rings.

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Note: Figure translations are in progress. See original paper for figures.