

# ON THE EXISTENCE OF MAPPINGS IN THE PLANE FOR DEGENERATING ELLIPTIC SYSTEMS OF FIRST ORDER

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**Abstract**

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*MATHEMATICS*

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## ON THE EXISTENCE OF MAPPINGS IN THE PLANE FOR DEGENERATING ELLIP- TIC SYSTEMS OF FIRST ORDER

*(Presented by Academician M. A. Lavrent'ev on 11 VIII 1969)*

In the present paper we consider the question of the existence of homeomorphic mappings of one domain onto another for the system

$$a_{11}f_{11} + a_{12}f_{12} = f_{22}, \quad a_{21}f_{11} + a_{22}f_{12} = -f_{21}, \quad (1)$$

where  $f_{ik} = \partial f_i / \partial x_k$ ;  $a_{ik}$  are functions of the variables  $x_1$  and  $x_2$ , for which in the domain  $D$  under consideration the ellipticity condition is satisfied

$$A = a_{11}a_{22} - \left( \frac{a_{12} + a_{21}}{2} \right)^2 > 0.$$

For the case of uniformly elliptic systems with one and with two pairs of characteristics, this question was studied in the works of M. A. Lavrent'ev, Z. Ya. Shapiro, B. V. Boyarskii, and others.

To establish the existence theorems formulated below, we use an approximation of the degenerating system (1) by means of nondegenerating systems, or else the domain  $D$  is approximated by domains in which the system (1) does not degenerate. Then the known existence theorems for the nondegenerate case are applied, and the required mapping is obtained by a limiting passage using results from the works <sup>(1,2)</sup>.

To simplify the exposition, we shall assume that the coefficients  $a_{ik}$  are continuous functions.

Let  $y = f(x)$  be a homeomorphic mapping of the domain  $D$  onto the domain  $\Delta$  in the  $n$ -dimensional Euclidean space  $E^n$ , where  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$ ,  $f(x) = (f_1(x), \dots, f_n(x))$ . Suppose that the integral

$$I(f, D, F) = \int_D F \left( x, f, \frac{df}{dx} \right) dx,$$

is bounded, where  $F(x, y, Z) = u^n(x, y)\|Z\|^n$ ;  $u(x, y)$  is a continuous and positive function in  $D \times \Delta$ ;  $Z = (z_{ij})$  is an  $n \times n$  matrix,

$$\|Z\| = \left( \sum_{i,j=1}^n z_{ij}^2 \right)^{1/2};$$

$dx$  is the volume element;  $df/dx \equiv f'_x = (\partial f_i / \partial x_j)$  is an  $n \times n$  matrix whose entries are partial derivatives understood in the sense of S. L. Sobolev. In this case the function  $y = f(x)$  will be differentiable almost everywhere in  $D$ .

At points of differentiability of the function  $f(x)$  we have  $dy = f'_x dx$ , where  $dy$  and  $dx$  are column vectors composed of the differentials  $dy_i$  and  $dx_i$ . For fixed  $x$ , the matrix  $f'_x$  transforms the sphere  $|dx| = 1$  into an ellipsoid; we denote the ratio of the largest axis of this ellipsoid to the smallest by  $Q(f, x)$ . Then the inequality

$$\|f'_x\|^n \leq n^{n/2} Q^{n-1}(f, x) |f'_x|$$

holds.

where  $|f'_x|$  is the determinant of the matrix  $f'_x$  (we assume that  $|f'_x| > 0$  almost everywhere in  $D$ ).

Denote by  $|D|$  and  $|\widetilde{D}|$  the Euclidean volume of the domain  $D$  and the volume of the domain  $D$  in the spherical metric. Suppose that there exists a function  $q(x)$ , continuous in the domain  $D$ , such that  $q(x) \geq Q(f, x)$  almost everywhere in the domain  $D$ . It is easy to verify that the inequalities

$$I(f, D, F_1) \leq n^{n/2} |\widetilde{\Delta}|, \quad I(f^{-1}, \Delta, \Phi_1) \leq n^{n/2} |\widetilde{D}|, \quad (2)$$

$$I(f, D, F_2) \leq n^{n/2} |\Delta|, \quad I(f^{-1}, \Delta, \Phi_2) \leq n^{n/2} |D|, \quad (3)$$

hold, where

$$F_1(x, y, Z) = (1 + |y|^2)^{-n} q^{-n+1}(x) \|Z\|^n, \quad \Phi_1(y, x, Z) = (1 + |x|^2)^{-n} q^{-n+1}(x) \|Z\|^n,$$

$$F_2(x, y, Z) = q^{-n+1}(x) \|Z\|^n, \quad \Phi_2(y, x, Z) = q^{-n+1}(x) \|Z\|^n.$$

Relations (2) and (3) are fundamental in applying the results of works <sup>(1,2)</sup> to the study of the properties of solutions of degenerate systems of the form (1).

Below, everywhere only mappings in the plane are considered ( $n = 2$ ).

For every homeomorphic mapping  $y = f(x)$  that is a solution of system (1), the inequality

$$Q(f, x) \leq A^{-1} \|C\| \cdot \|H\|$$

holds, where

$$C = \begin{pmatrix} a_{11} & \frac{a_{12} + a_{21}}{2} \\ \frac{a_{12} + a_{21}}{2} & a_{22} \end{pmatrix}, \quad H = \begin{pmatrix} a_{11}a_{22} - a_{12}a_{21} & \frac{a_{12} - a_{21}}{2} \\ \frac{a_{12} - a_{21}}{2} & 1 \end{pmatrix},$$

and therefore relations (2), (3) will be satisfied if one sets  $q(x) = A^{-1} \|C\| \cdot \|H\|$ .

Let us first consider a simpler case, when (1) is the Beltrami system ( $a_{12} = a_{21}$ ,  $a_{11}a_{22} - a_{12}^2 = 1$ ) and its solutions are mappings with one pair of characteristics.

**Theorem 1.** *Let  $D$  and  $\Delta$  be arbitrary simply connected domains whose boundaries contain more than two points. Fix arbitrary points  $e_1$  and  $e_1^*$  of the domains  $D$  and  $\Delta$ , respectively, and  $e_2, e_2^*$  arbitrary simple ends of the boundaries of the domains  $D$  and  $\Delta$ . Suppose that in some subdomain  $g \subset D$ , into which the simple end  $e_2$  enters, system (1) is nondegenerate, i.e.  $\sup q(x) < +\infty$ , where the supremum is taken over  $x \in g$ .*

*Then there exists a solution  $y = f(x)$  of system (1) that realizes a homeomorphism of the domain  $D$  onto the domain  $\Delta$  and is such that  $f(e_i) = e_i^*$  ( $i = 1, 2$ ).*

For the proof of this assertion, the property of invariance of solutions of the Beltrami system under conformal transformations is essential. System (1), in the general case, does not possess an analogous property, and, in particular, this is connected with the difficulties in proving theorems on the existence of mappings with two pairs of characteristics for the degenerating system (1).

Consider the case of system (1) in which its solutions realize mappings that send infinitesimally small circles of the plane ( $x$ ) into infinitesimally small ellipses of the plane ( $y$ ). In this case  $a_{11} = a_{22}$ ,  $a_{12} = -a_{21}$ , and

$$|f'_x| = a_{11} |\nabla f_1|^2 = a_{11} (a_{11}^2 + a_{12}^2)^{-1} |\nabla f_2|^2.$$

Therefore

$$\int_D \frac{a_{11} |\nabla f_1|^2}{(1 + |f|^2)^2} dx = |\tilde{\Delta}|, \quad \int_D \frac{a_{11}}{a_{11}^2 + a_{12}^2} \frac{|\nabla f_2|^2}{(1 + |f|^2)^2} dx = |\tilde{\Delta}|, \quad (4)$$

$$\int_D a_{11} |\nabla f_1|^2 dx = |\Delta|, \quad \int_D \frac{a_{11}}{a_{11}^2 + a_{12}^2} |\nabla f_2|^2 dx = |\Delta|. \quad (5)$$

We shall denote by  $|e|$  the body of the simple end  $e$ , and by  $\partial\Delta$  the boundary of the domain  $\Delta$ .

**Theorem 2.** Let a simply connected domain  $\Delta$  with finite area  $|\Delta|$  lie in the half-plane  $y_2 > 0$ , and let its boundary contain the rectilinear segment

$$L = \{y : \alpha_1 \leq y_1 \leq \alpha_2, y_2 = 0\}.$$

Denote  $a_i = (\alpha_i, 0)$  ( $i = 1, 2$ ), and suppose that

$$\lim_{\substack{y \rightarrow a_1 \\ y \in \partial\Delta \setminus L}} \frac{y_1 - \alpha_1}{|y - a_1|} > -1, \quad \lim_{\substack{y \rightarrow a_2 \\ y \in \partial\Delta \setminus L}} \frac{y_1 - \alpha_2}{|y - a_2|} < 1.$$

By  $\gamma$  denote the smallest closed arc of prime ends containing the set  $L \setminus (a_1 \cup a_2)$ . Fix on the boundary of the domain  $\Delta$  a prime end  $e_3^*$  not belonging to  $\gamma$ . Let  $D$  be an arbitrary simply connected domain on whose boundary three distinct prime ends  $e_i$  ( $i = 1, 2, 3$ ) are given.

Suppose that the set of prime ends at which system (1) degenerates is contained in the closed arc of prime ends  $\widehat{e_1 e_2}$ , not containing the prime end  $e_3$ . Let  $a_{11} = a_{22}$ ,  $a_{12} = -a_{21}$ , and let the function  $(a_{11}^2 + a_{12}^2)a_{11}^{-1}$  be bounded in the domain  $D$ .

Then there exists a homeomorphic mapping  $y = f(x)$  of the domain  $D$  onto the domain  $\Delta$ , which is a solution of system (1), such that  $f(e_3) = e_3^*$  and the arc of prime ends  $\widehat{e_1 e_2}$  under the mapping  $y = f(x)$  passes into the arc of prime ends  $\gamma$ .

**Theorem 3.** Consider an arbitrary simply connected domain  $D$  and the strip

$$\Delta = \{y : 0 < y_2 < 1\}.$$

Let  $e_i$  ( $i = 1, 2, 3$ ) be three distinct prime ends of the boundary of the domain  $D$ ; let  $e_1^*, e_2^*$  be prime ends of the domain  $\Delta$ , whose impression is  $\infty$ ; and let  $e_3^*$  be some point of the set  $\partial\Delta$ . Suppose that in some neighborhoods of the prime ends  $e_i$  ( $i = 1, 2, 3$ ) system (1) does not degenerate, the function  $(a_{11}^2 + a_{12}^2)a_{11}^{-1}$  is bounded in the domain  $D$ , and  $a_{11} = a_{22}$ ,  $a_{12} = -a_{21}$ .

Then there exists a homeomorphic mapping  $y = f(x)$  of the domain  $D$  onto the strip  $\Delta$ , which is a solution of system (1), and such that  $f(e_i) = e_i^*$  ( $i = 1, 2, 3$ ).

Let there be given an elliptic differential equation

$$\sum_{i,k=1}^2 b_{ik} v_{ik} + \sum_{i=1}^2 c_i v_i = 0, \quad (6)$$

where  $v_{ik} = \partial^2 v / \partial x_i \partial x_k$ ,  $v_i = \partial v / \partial x_i$ . We shall assume that the coefficients  $b_{ik} = b_{ki}$ ,  $c_i$  are functions continuous in the domain  $\overline{D}$ . If the function  $y = f(x)$  satisfies system (1), then  $y_2 = f_2(x)$  is a solution of some differential equation. Let this equation be equation (6).

We shall say that a barrier is defined at a point  $z \in \partial D$  if there exists a generalized superharmonic (see (3), p. 340) function  $w(x, z)$  of the variable  $x$ , relative to equation (6), continuous in  $\bar{D}$ , positive in  $\bar{D}$  everywhere except the point  $z$ , and  $w(z, z) = 0$ .

If the barrier  $w(x, z)$  is defined for all  $z$  from some set  $H \subset \partial D$ , and the function  $w(x, z)$  is uniformly continuous with respect to  $x$  relative to  $z \in H$ , then  $w(x, z)$  will be called a uniform barrier on the set  $H$ .

**Theorem 4.** Let  $D$  be a simply connected domain whose boundary contains more than one point, and let  $G$  be the strip  $\{y : 0 < y_2 < 1\}$ . Let the prime ends  $e_1$  and  $e_2$  of the boundary of the domain  $D$  divide this boundary into two open arcs of prime ends  $\gamma_1$  and  $\gamma_2$ . Suppose that for each prime end  $e \in \gamma_i$  ( $i = 1, 2$ ) there is a principal point  $a$  of the prime end such that at this point system (1) does not degenerate, or for this point there exists a barrier with respect to equation (6).

Then there exists a homeomorphic mapping  $y = f(x)$  of the domain  $D$  onto a domain  $\Delta \subset G$ , which is a solution of system (1), and  $f(\gamma_i) \subset \gamma_i^*$  ( $i = 1, 2$ ), where

$$\gamma_i^* = \{y : y_2 = \delta_i\}, \quad \delta_1 = 0, \quad \delta_2 = 1.$$

We give a sufficient condition for the existence of a barrier at the principal point  $a$  of a simple end  $e$  of the domain  $D$ . Suppose that there exists a subdomain  $g \subset D$ , into which the simple end  $e$  enters, such that the coefficients of equation (6) are bounded in  $g$  and the domain  $g$  lies on one side of a straight line parallel to the  $ox_1$  axis and passing through the point  $a$ . Let  $a$  not belong to the bodies of the other simple ends of the domain  $D$ . If

$$\overline{\lim}_{x \rightarrow a, x \in g} \left[ c_2 - \frac{v_{22}}{x_2} \right] < 0,$$

then a barrier exists at the point  $a$ . Moreover, the function  $w(x, a) = (x_1 - a_1)^2 + (x_2 - a_2)^\beta$ , where  $\beta$  is some number from the interval  $(0, 1)$ , may be taken as the barrier.

**Example.** Consider the system

$$x_2 u_{x_1} = v_{x_2}, \quad x_2 u_{x_2} = -v_{x_1},$$

which describes an axisymmetric flow of an ideal fluid. In this case equation (6) takes the form

$$x_2 \Delta v - v_{x_2} = 0.$$

Let the domain  $D$  lie in the upper half-plane and be a curvilinear strip bounded by two curves  $\gamma_i$  ( $i = 1, 2$ ). Then all the conditions of Theorem 4 are fulfilled,

and there exists a mapping  $y = f(x)$  with the properties indicated in that theorem.

**Remark 1.** The curvilinear strip  $D$  contains two simple ends  $e_i$  ( $i = 1, 2$ ), whose bodies are  $\infty$ . The possible behavior of the mapping  $y = f(x)$  of the domain  $D$  onto the domain  $\Delta \subset \bar{G}$  as  $x \rightarrow e_i$  ( $i = 1, 2$ ) is described in Theorems 4 and 5 of paper <sup>2</sup>, taking into account the relations (4). Using these theorems, as well as the properties of a regular barrier, one can obtain the following result.

Let there exist subdomains  $g_i$  ( $i = 1, 2$ ) of the domain  $D$  such that the simple end  $e_i$  enters  $g_i$  ( $i = 1, 2$ ), and the domain  $g_i$  can be enclosed between two parallel straight lines, with  $\rho(\gamma_1 \cap \bar{g}_i, \gamma_2 \cap \bar{g}_i) > 0$ , if one of the parallel straight lines between which the domain  $g_i$  lies is contained in the half-plane  $x_2 \leq 0$  (here  $\bar{g}_i$  is the closure of the set  $g_i$ ). Then, in the example under consideration, the domain  $\Delta$  coincides with the strip  $G$ .

**Remark 2.** Solutions of system (1) can be represented as a superposition of two solutions: the first maps infinitely small ellipses into infinitely small circles, and the second maps infinitely small circles into infinitely small ellipses. In accordance with this, theorems may be formulated which combine, on the one hand, Theorem 1 and, on the other hand, one of Theorems 2, 3, 4.

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## CITED LITERATURE

<sup>1</sup> I. S. Ovchinnikov, DAN, **187**, No. 1 (1969). <sup>2</sup> I. S. Ovchinnikov, DAN, **190**, No. 2 (1970). <sup>3</sup> R. Courant, *Equations with Partial Derivatives*, Moscow, 1964.

*Note: Figure translations are in progress. See original paper for figures.*

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