ANALYTICAL SOLUTION OF THE PROBLEM OF CONTROL OF A SPATIAL ROTATIONAL MANEUVER

Academician B. N. PETROV, V. A. BODNER, K. B. ALEKSEEV

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Fig. 1

Figure 1: Fig. 1

Abstract

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CYBERNETICS AND CONTROL THEORY

Academician B. N. PETROV, V. A. BODNER, K. B. ALEKSEEV

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1. Control of the attitude of flying vehicles by means of a single rotation about a certain axis [1] has better potential capabilities in comparison with the commonly used three successive rotations about orthogonal axes fixed to the vehicle. This note gives an analytical solution of the problem of synthesizing an attitude-control algorithm. For this purpose the principle of extensive control is applied, consisting in the choice of a control-moment vector, bounded in magnitude,

$$\mathbf{M} = \|M_1,\ M_2,\ M_3\|^T$$

from the condition of motion of the vehicle relative to a prescribed axis when the components of the moment M_i are applied along the axes fixed to the vehicle.

2. Introduce three coordinate systems with origin at the center of mass of the vehicle: the inertial system X_i (here and below $i = 1, 2, 3$), the system x_i fixed to the vehicle, and the auxiliary system ξ_i. The angular position of the vehicle in the system X_i is determined by the Euler angles ψ_i (Fig. 1). The transformation matrix $X_i \to x_i$ has the form

$$\alpha = \|\alpha_{ij}\|, \tag{1}$$

where α_{ij} are functions of the angles ψ_i.

Fig. 1

The angle φ of rotation of the vehicle about an axis specified by the unit vector e_φ, and the orientation of this vector, are determined by

$$\varphi = \arccos(\operatorname{Sp}\alpha - 1)/2, \qquad e_\varphi = \|\nu_1, \nu_2, \nu_3\|^T, \tag{2}$$

where

$$\nu_i = \frac{\alpha_{i+1,i+2} - \alpha_{i+2,i+1}}{2\sin\varphi}, \qquad \operatorname{Sp}\alpha = \sum_{i=1}^{3} \alpha_{ii}$$

is the trace of the matrix α.

Let us make the axis ξ_1 of the auxiliary system coincide with the vector e_φ. Denote the unit vectors of the axes ξ_i by e_i, with $e_1 = e_\varphi$. The transformation matrix $x_i \to \xi_i$ has the form

$$\nu = \|\nu_{ij}\|, \tag{3}$$

where $\nu_{i1} = \nu_i$, and $\nu_{12}, \dots, \nu_{33}$ are any numbers for which the matrix α is orthogonal.

The inertia tensor I_ξ of the vehicle in the system ξ_i will be

$$I_\xi = \nu I_x \nu^T, \tag{4}$$

where

$$I_x = \begin{Vmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{Vmatrix},$$

and J_1, J_2, J_3 are the principal moments of inertia of the vehicle.

Let the vector of angular velocity of the vehicle ω coincide with the vector e_φ, i.e.

$$\omega = \dot\varphi e_\varphi. \tag{5}$$

When this condition is fulfilled, the vector of angular momentum of the vehicle will be

$$\mathbf{K} = J_k \dot\varphi e_k, \tag{6}$$

where

$$J_\kappa^2 = \sum_{i=1}^{3} J_i^2 \nu_i^2, \qquad \mathbf{e}_\kappa = \sum_{i=1}^{3} \gamma_i \mathbf{e}_i$$

is a unit vector, $\gamma_i = J_i \nu_i / J_\kappa$ are the direction cosines, and $\mathbf{e}_i$ are unit vectors of the axes x_i.

It follows from the expression for γ_i that the angular-momentum vector $\mathbf{K}$ of a dynamically asymmetric vehicle does not coincide with the axis $\mathbf{e}_\varphi$. Let us determine the components of this vector in the direction of the axis $\mathbf{e}_\varphi$ and in the direction $\mathbf{e}_\vartheta$, perpendicular to the axis of rotation:

$$K_\varphi = \mathbf{K} \cdot \mathbf{e}_\varphi = J_\varphi \dot{\varphi}, \qquad K_\vartheta = \mathbf{K} \cdot \mathbf{e}_\vartheta = J_\vartheta \dot{\varphi}, \qquad J_\varphi = J_{11} = \sum_{i=1}^{3} J_i \nu_i^2,$$

$$J_\vartheta^2 = \sum_{i=1}^{3} (J_{i+1} - J_{i+2})^2 \nu_{i+1}^2 \nu_{i+2}^2. \tag{7}$$

The equation of motion of the vehicle in Euler form, taking (5) into account, will be

$$d\mathbf{K}/dt + \dot{\varphi}\, \mathbf{e}_\varphi \times \mathbf{K} = \mathbf{M}. \tag{8}$$

The sought vector of the control torque $\mathbf{M}$ must ensure rotation of the vehicle about the prescribed axis $\mathbf{e}_\varphi$. Transforming (8), taking into account (6) and (7), we obtain

$$J_\kappa \ddot{\varphi}\, \mathbf{e}_\kappa + J_\vartheta \dot{\varphi}^2 \mathbf{e}_\Gamma = \mathbf{M}, \qquad \mathbf{e}_\Gamma = \mathbf{e}_\varphi \times \mathbf{e}_\kappa / |\mathbf{e}_\varphi \times \mathbf{e}_\kappa|. \tag{9}$$

It follows from this that the vector $\mathbf{M}$ must provide the required law of motion $\mathbf{M}_1 = J_\kappa \ddot{\varphi}\, \mathbf{e}_\varphi$ of the vehicle about the axis $\mathbf{e}_\varphi$ and compensate the gyroscopic torque $\mathbf{M}_2 = J_\vartheta \dot{\varphi}^2 \mathbf{e}_\Gamma$ arising in this motion. The modulus of the vector $\mathbf{M}$ is equal to

$$M = J_\kappa |\ddot{\varphi}| \left(1 + J_\vartheta^2 \dot{\varphi}^4 / J_\kappa^2 \ddot{\varphi}^2\right)^{1/2}. \tag{10}$$

We specify the orientation of the vector $\mathbf{M}$ in the body-fixed coordinate system by three direction cosines $\beta_1, \beta_2, \beta_3$. If M_{mi} is the maximum admissible value of the torques along the body-fixed axes, then the components of the torques will be

$$M_i = M_{mi}, \qquad M_{i+1} = M_{mi} |\beta_{i+1}| / \beta_i^*, \qquad M_{i+2} = M_{mi} |\beta_{i+2}| / \beta_i^*, \tag{11}$$

where β_i^* is the maximum value of the direction cosine of the vector $\mathbf{M}$ in the system x_i.

The torques M_i along the body-fixed axes must be chosen so that the vehicle performs a maneuver about the axis $\mathbf{e}_\varphi$ through the angle φ in time T, with

$$\varphi(0) = -\varphi_0, \qquad \dot{\varphi}(0) = 0, \qquad \varphi(T) = \dot{\varphi}(T) = 0, \tag{12}$$

at the minimum possible energy expenditure Q

$$Q = M_{mi} \int_0^T \left(1 + \frac{|\beta_{i+1}| + |\beta_{i+2}|}{\beta_i^*}\right) dt. \tag{13}$$

3. Introducing into consideration the angle θ between the vectors of the control torque $\mathbf{M}$ and of the angular momentum $\mathbf{K}$, we write equation (9) in scalar form:

$$\ddot{\varphi} = \pm \frac{M}{J_\kappa} \cos\theta, \qquad \dot{\varphi}^2 = \frac{M}{J_\vartheta} \sin\theta, \tag{14}$$

whence

$$\operatorname{tg}\theta = J_\vartheta \dot{\varphi}^2 / J_\kappa \ddot{\varphi}. \tag{15}$$

To obtain the direction cosines β_i, we project equation (8) onto the axes of the body-fixed system:

$$J_i \nu_i \ddot{\varphi} + (J_{i+2} - J_{i+1}) \nu_{i+1} \nu_{i+2} \dot{\varphi}^2 = M_i \tag{16}$$

and use expressions (10) and (15); then

$$\beta_i = (\tilde{a}_i + \tilde{b}_i \operatorname{tg}\theta) \cos\theta, \qquad \tilde{a}_i = (J_i \nu_i / J_\kappa) \operatorname{sign} \ddot{\varphi},$$

$$\tilde{b}_i = (J_{i+2} - J_{i+1}) \nu_{i+1} \nu_{i+2} / J_\vartheta. \tag{17}$$

Taking (17) into account, equations (14) take the form

$$\begin{aligned} \ddot{\varphi} &= \pm M_{mi} / J_{\text{k}} (\tilde{a}_i + \tilde{b}_i \operatorname{tg}\theta), \\ \dot{\varphi}^2 &= M_{mi} \operatorname{tg}\theta / J_{\text{e}} (\tilde{a}_i + \tilde{b}_i \operatorname{tg}\theta). \end{aligned} \tag{18}$$

The choice of the current value of the angle θ must be made from the condition of compatibility of equations (18), which reduces to the equation

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

$$d\theta/\cos^2\theta\sqrt{(\tilde{a}_i+\tilde{b}_i\,\mathrm{tg}\,\theta)\,\mathrm{tg}\,\theta}=(2\sqrt{M_{mi}J_{\mathrm{e}}}/a_{iJ_{\mathrm{k}}})\,d\tau, \tag{19}$$

and whose solution can be represented in the form

$$\mathrm{tg}\,\theta=\begin{cases}(a_i/b_i)\sin^2 nt, & \text{for } \beta_i^*=(a_i-b_i\,\mathrm{tg}\,\theta)\cos\theta,\\ (a_i/b_i)\,\mathrm{sh}^2\,nt, & \text{for } \beta_i^*=(a_i+b_i\,\mathrm{tg}\,\theta)\cos\theta,\end{cases} \tag{20}$$

where $a_i=|\tilde{a}_i|,\ b_i=|\tilde{b}_i|,\ n=\sqrt{M_{mi}b_{iJ_{\mathrm{e}}}/J_{\mathrm{k}}a_i}$.

Substituting the value of $\mathrm{tg}\,\theta$ from (20) into (18), we obtain

$$\ddot{\varphi}=\begin{cases}m/\cos^2 nt, & \text{for } \beta_i^*=(a_i-b_i\,\mathrm{tg}\,\theta)\cos\theta,\\ m/\mathrm{ch}^2\,nt, & \text{for } \beta_i^*=(a_i+b_i\,\mathrm{tg}\,\theta)\cos\theta,\end{cases}\qquad m=M_{mi}/J_{\mathrm{k}}a_i. \tag{21}$$

Equation (21), taking into account the boundary conditions (12), characterizes the dynamic properties of the vehicle under extensive control. For the synthesis of the extensive-control system we shall apply Pontryagin' s maximum principle (^{2}).

4. The switching time $t_1 \in (0,T)$ of the control torque of maximum magnitude is found from the solution of equations (21) and 12.

Fig. 2

Fig. 3

Without loss of generality, let us assume that $\tilde{a}_i$ and $\tilde{b}_i$ have the same sign; therefore the solution of equation (21) for $t \le t_1$ will be

$$\dot{\varphi}=(m/n)\,\mathrm{tg}\,nt,\qquad \varphi=-\varphi_0+(m/n^2)\ln\mathrm{ch}\,nt. \tag{22}$$

At $t=t_1$, when $\theta_1=\mathrm{arc\,tg}((a_i/b_i)\,\mathrm{sh}^2\,nt_1)$, the control action is switched. In this case the component $M_1=J_{\mathrm{k}}\ddot{\varphi}$ of the torque $\mathbf{M}$ changes in magnitude and direction, whereas the direction of the component $M_2=J_{\mathrm{e}}\dot{\varphi}^2$ remains unchanged.

From the condition of compensation of the gyroscopic torque

$$M_2 = J_{\text{e}}\dot{\varphi}^2 = M_m \sin\theta \tag{23}$$

for $M_m = M_{mi}/(a_i + b_i \operatorname{tg}\theta_1)\cos\theta_1$ or $M_m = M_{mi}/(a_i - b_i \operatorname{tg}\theta_2)\cos\theta_2$, we obtain

$$\operatorname{tg}\theta_2 = \frac{a_i}{b_i}\,\frac{\operatorname{sh}^2 nt_1}{1 + 2\operatorname{sh}^2 nt_1}. \tag{24}$$

Determination of the angle θ for $t > t_1$, taking into account the initial value $\theta_2(t) = \theta_2$, is performed from the expression

$$\operatorname{tg}\theta = (a_i/b_i)\sin^2[n(t - t_1) - \chi], \tag{25}$$

where $\chi = \operatorname{arc\,tg}(\operatorname{th} nt_1)$. Since $\theta(T) = 0$, from (25) we find

$$\operatorname{tg} n(T - t_1) = \operatorname{th} nt_1. \tag{26}$$

To obtain the second equation relating the sought quantities T^* and t_1, let us solve the equation of motion (21) for $t_1 \le t \le T$. In this case

$$\ddot{\varphi} = -\frac{m}{\cos^2[n(t - t_1) - \chi]}, \qquad \dot{\varphi} = \dot{\varphi}_1 - \frac{m}{n}\{\operatorname{tg}[n(t - t_1) - \chi] + \operatorname{th} nt_1\},$$

$$\varphi = \varphi_1 + \left(\dot{\varphi}_1 - \frac{m}{n}\operatorname{th} nt_1\right)(t - t_1) + \frac{m}{n^2}\left\{\ln\frac{\cos[n(t - t_1) - \chi]}{\cos\chi}\right\}. \tag{27}$$

Putting $t = T$ in (27) and using (12), we find

$$\frac{\operatorname{ch} nt_1}{\cos n(T - t_1)} = \exp\left(\frac{n^2}{m}\varphi_0\right). \tag{28}$$

From (26) and (28), under the assumption that β_i^* is maximal throughout the entire control process, we obtain the switching time t_1 and the minimum time T_m for rotating the craft through the angle φ_0:

$$t_1 = (1/n)\operatorname{arc\,ch}\lambda, \qquad T_m = (1/n)(\operatorname{arc\,tg}\lambda + \operatorname{arc\,th}\lambda),$$

$$\lambda^2 = \operatorname{th}(n^2/m)\varphi_0. \tag{29}$$

Fig. 4

Figure 4: Fig. 4

The expressions for the moments about the axes of the craft that provide a rotation in minimum time about the axis e_φ have the form

$$M_i = M_{mi},$$

$$M_{i+1} = M_{mi} \begin{cases} \dfrac{\tilde{a}_{i+1}}{a_i} \operatorname{ch}^{-2} nt + \dfrac{\tilde{b}_{i+1}}{b_i} \operatorname{th}^2 nt, & \text{for } 0 \le t \le t_1, \\ \dfrac{\tilde{a}_{i+1}}{a_i} \cos^{-2}[n(t-t_1)-\chi] + \dfrac{\tilde{b}_{i+1}}{b_i} \operatorname{tg}^2[n(t-t_1)-\chi], & \text{for } t_1 \le t \le T. \end{cases} \tag{30}$$

The expenditure of working substance for control in accordance with (13) and (30) will be

$$Q = M_m \left\{ T + \sum_{i=1}^{3} \left[\frac{\tilde{a}_{i+1}}{a_i} \operatorname{th} nt_1 + \frac{\tilde{b}_{i+1}}{b_i} \left(t_1 + \frac{a_i \operatorname{th} nt_1}{nb_i(1+\operatorname{th} nt_1)} \right) \right] \right\}, \tag{31}$$

where M_m is the maximum moment, taken to be the same about the three axes.

In Fig. 2 the trajectory of motion of the representative point on the phase plane is given; in Fig. 3 the character of the change in the modulus of the vector of the control moment is shown.

Transferring the craft from the initial $\varphi(0) = -\varphi_0$, $\dot{\varphi}(0) = 0$ to the final position $\varphi(T) = \dot{\varphi}(T) = 0$ in a specified time T_3 requires two switchings of the control moment M_m, maximal in magnitude. If the switching instants are denoted by t_1 and $t_1 + t_2$, with $T_3 = t_1 + t_2 + t_3$, where t_3 is the braking time of the craft (Fig. 4), and if the method of investigation described above is applied, then we obtain equations for determining t_1, t_2, and t_3:

Fig. 4

$$T_3 = t_1 + t_2 + t_3, \qquad \operatorname{th} nt_1 = \operatorname{tg} nt_1, \tag{32}$$

$$\operatorname{ch} nt_1 / \cos nt_3 = \exp[(n^2/m)\varphi_0 - nt_2 \operatorname{th} nt_1],$$

The implementation of the algorithm of optimal extensive control must be carried out on the onboard computing device, in which the direction of the rotation axis, the magnitude and sign of the resultant rotation angle are determined, and the control moments along the body-fixed axes are formed. In this case either the minimum time of the turning maneuver or the minimum expenditure of working substance over the specified time can be ensured.

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CITED LITERATURE

1. J. Synge, *Classical Dynamics*, Moscow, 1963.

2. L. S. Pontryagin et al., *The Mathematical Theory of Optimal Processes*, Moscow, 1961.

Note: Figure translations are in progress. See original paper for figures.